Published by Institute of Physics Publishing for SISSA

Received: *March 2, 2005*
Revised: *May 2, 2005*
Accepted: *May 3, 2005*
Published: *June 14, 2005*

Matching of singly- and doubly-unresolved limits of tree-level QCD squared matrix elements

Gábor Somogyi and Zoltán Trócsányi

University of Debrecen and
Institute of Nuclear Research of the Hungarian Academy of Sciences
H-4001 Debrecen, PO Box 51, Hungary
E-mail: somogyga@dragon.unideb.hu, z.trocsanyi@atomki.hu

Vittorio Del Duca

Istituto Nazionale di Fisica Nucleare, Sez. di Torino
via P. Giuria, 1 - 10125 Torino, Italy
E-mail: delduca@to.infn.it

Abstract: We describe how to disentangle the singly- and doubly-unresolved (soft and/or collinear) limits of tree-level QCD squared matrix elements. Using the factorization formulae presented in this paper, we outline a viable general subtraction scheme for computing next-to-next-to-leading order corrections for electron-positron annihilation into jets.

Keywords: Jets, QCD.

JHEP06(2005)024

© SISSA 2005

Contents

JHEP06(2005)024

1. Introduction

QCD, the theory of strong interactions, is an important component of the Standard Model of elementary particle interactions. It is asymptotically free, which allows us to compute cross sections of elementary particle interactions at high energies as a perturbative expansion in the running strong coupling $\alpha_s(\mu_R)$. However, the running coupling $\alpha_s(\mu_R)$ remains rather large at the energies relevant at recent and future colliders. In addition, to leading order in the perturbative expansion, the coupling varies sizeably with the choice of the (unphysical) renormalisation scale μ_R. In hadron-initiated processes, the situation above is worsened by the dependence of the cross section on the (also unphysical) factorisation scale μ_F, which separates the long-distance from the short-distance part of the strong interaction. Thus, a leading-order evaluation of the cross section yields rather unreliable predictions for most processes in the theory. To improve this situation, in the past 25 years the radiative corrections at the next-to-leading order (NLO) accuracy have been computed. These efforts have culminated, when process-independent methods were presented for computing QCD cross sections to NLO accuracy, namely the slicing [1, 2], subtraction [3 − 5] and dipole subtraction [6] methods. In some cases, though, the NLO corrections were found to be disturbingly large, and/or the dependence on μ_R (and eventually μ_F) was found to be still sizeable, thus casting doubts on the applicability of the perturbative predictions. When the NLO corrections are found to be of the same order as the leading-order prediction, the only way to assess the reliability of QCD perturbation theory is the computation of the next-to-next-to-leading order (NNLO) corrections.

In recent years severe efforts have been made to compute the NNLO corrections to the parton distribution functions [7] and important basic processes, such as vector boson production [8 − 11] and Higgs production [9, 12 − 14] in hadron collisions and jet production in electron-positron annihilation [15 − 17]. These computations evaluate also the phase space integrals in d dimensions, thus, do not follow the process-independent methods used to compute the NLO corrections. Presently it is not clear whether those techniques [18 − 21] can be directly applied for processes with more complex final states.

The more traditional approach relies on defining approximate cross sections which match the singular behaviour of the QCD cross sections in all the relevant unresolved limits. Various attempts were made in this direction in refs. [22 − 30]. In general, the definition of the approximate cross sections must rely on the single and double unresolved limits of the QCD squared matrix elements. Although the infrared limits of QCD matrix elements have been extensively studied [31 − 34], the formulae presented in the literature do not lend themselves directly for devising the approximate cross sections for two reasons. The first problem is that the various single and double soft and/or collinear limits overlap in a very

JHEP06(2005)024

complicated way and the infrared factorization formulae have to be written in such forms that these overlaps can be disentangled so that double subtraction is avoided. The second problem is that even if the factorization formulae are written such that double subtraction does not happen, the expressions cannot straightforwardly be used as subtraction formulae, because the momenta of the partons in the factorized matrix elements are unambiguously defined only in the strict soft and collinear limits. In order to define the approximate cross sections one also has to factorize the phase space of the unresolved partons such that the singular factors can be integrated and the remaining expressions can be combined with the virtual correction leading to cross sections which are finite and integrable in four dimensions.

In this paper we present a solution to the first problem. Due to momentum conservation constraints the expressions presented here are not unique. Nevertheless, these expressions may be useful in making the second step leading to a general subtraction scheme for computing QCD cross sections at the NNLO accuracy. We outline such a scheme for processes without coloured partons in the initial state, but do not explicitly define the approximate cross sections, which we leave for later work.

The paper is organised as follows: after setting the notation in section 2, we review in section 3 the singly-unresolved limits and the subtraction terms relevant to NLO cross sections. In section 4 we review the doubly-unresolved limits and introduce the corresponding subtraction term. In section 5 we introduce the subtraction term that regularizes the squared matrix element in all the unresolved regions of the phase space relevant to NNLO computations. That subtraction term is the keystone of this paper. In sections 6 and 7 we derive the iterated singly-unresolved limits and the singly-unresolved limits of the doubly-unresolved factorization formulae, respectively. Those singly-unresolved limits are used in the construction of the several contributions to the subtraction term presented in section 5. In section 8 we sketch the proof of the validity of the subtraction term of section 5 as a correct regulator of the divergences that occur in all the unresolved regions of the phase space relevant to NNLO computations. In section 9 we outline a possible general subtraction method for computing NNLO corrections to jet cross sections. In section 10 we draw our conclusions.

2. Notation

We consider matrix elements of processes with $m+2$ coloured particles (partons) in the final-state. Any number of additional non-coloured particles is allowed, too, but they will be suppressed in the notation. Resolved partons will be labelled by $i, j, k, \dots$, unresolved ones are r and s.

The colour indices of the partons are denoted by c_i, which range over $1, \dots, N_c^2-1$ for gluons (or any other partons, such as gluinos, in the adjoint representation of the gauge group) and over $1, \dots, N_c$ for quarks and antiquarks (or any other partons, such as squarks, in the fundamental representation). Spin indices are generically denoted by s_i. As in ref. [6], we formally introduce an orthogonal basis of unit vectors $|c_1, \dots, c_m\rangle \otimes |s_1, \dots, s_m\rangle$ in the space of colour and spin, in such a way that an amplitude of a process involving m external

JHEP06(2005)024

partons, $\mathcal{M}_m^{\{c_I,s_I\}}(\{p_I\})$ with definite colour, spin and momenta $\{p_I\}$ can be written as

$$\mathcal{M}_m^{\{c_I,s_I\}}(\{p_I\}) \equiv \Big(\langle c_1,\ldots,c_m| \otimes \langle s_1,\ldots,s_m|\Big)|\mathcal{M}_m(\{p_I\})\rangle\,. \tag{2.1}$$

Thus $|\mathcal{M}_m\rangle$ is an abstract vector in colour and spin space, and its normalization is fixed such that the squared amplitude summed over colours and spins is

$$|\mathcal{M}_m|^2 = \langle\mathcal{M}_m||\mathcal{M}_m\rangle\,. \tag{2.2}$$

The matrix element has the following formal loop expansion:

$$|\mathcal{M}\rangle = |\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)}\rangle + |\mathcal{M}^{(2)}\rangle + \cdots\,, \tag{2.3}$$

where $|\mathcal{M}^{(0)}\rangle$ denotes the tree-level contribution, $|\mathcal{M}^{(1)}\rangle$ is the one-loop contribution, $|\mathcal{M}^{(2)}\rangle$ is the two-loop expression and the dots stand for higher-loop contributions. The amplitude $|\mathcal{M}\rangle$ is assumed to be renormalized. In this paper we study the infrared behaviour of the tree-level contribution, therefore, renormalization concerns us only to the extent of regularization. We use tree amplitudes obtained in conventional dimensional regularization.

Colour interactions at the QCD vertices are represented by associating colour charges $\boldsymbol{T}_i$ with the emission of a gluon from each parton i. The colour charge $\boldsymbol{T}_i = \{T_i^n\}$ is a vector with respect to the colour indices n of the emitted gluon and an $SU(N_c)$ matrix with respect to the colour indices of the parton i. More precisely, for a final-state parton i the action onto the colour space is defined by

$$\langle c_1,\ldots,c_i,\ldots,c_m|T_i^n|b_1,\ldots,b_i,\ldots,b_m\rangle = \delta_{c_1b_1}\ldots T^n_{c_ib_i}\ldots\delta_{c_mb_m}\,, \tag{2.4}$$

where T^n_{cb} is the colour-charge matrix in the representation of the final-state particle i, i.e. $T^n_{cb} = \mathrm{i}f_{cnb}$ if i is a gluon or a gluino, $T^n_{\alpha\beta} = t^n_{\alpha\beta}$ if i is a (s)quark and $T^n_{\alpha\beta} = -t^n_{\beta\alpha}$ if i is an anti(s)quark. Using this notation, we define the two-parton colour-correlated squared tree-amplitudes, $|\mathcal{M}^{(0)}_{(i,k)}|^2$, as

$$\begin{aligned}\left|\mathcal{M}^{(0)}_{(i,k)}(\{p_I\})\right|^2 &\equiv \langle\mathcal{M}^{(0)}(\{p_I\})|\,\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_k\,|\mathcal{M}^{(0)}(\{p_I\})\rangle \\ &= \Big[\mathcal{M}^{(0);\ldots a'_i\ldots a'_k\cdots}(\{p_I\})\Big]^*\,T^n_{a'_ia_i}\,T^n_{a'_ka_k}\,\mathcal{M}^{(0);\ldots a_i\ldots a_k\cdots}(\{p_I\})\,,\end{aligned} \tag{2.5}$$

and similarly the doubly two-parton colour-correlated squared tree-amplitudes $|\mathcal{M}^{(0)}_{(i,k),(j,l)}|^2$,

$$\left|\mathcal{M}^{(0)}_{(i,k)(j,l)}\right|^2 \equiv \langle\mathcal{M}^{(0)}|\,\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\}\,|\mathcal{M}^{(0)}\rangle\,, \tag{2.6}$$

where the anticommutator $\{\boldsymbol{T}_i\!\cdot\!\boldsymbol{T}_k,\boldsymbol{T}_j\!\cdot\!\boldsymbol{T}_l\}$ is non-trivial only if $i=j$ or $k=l$, see eq. (2.8).

In our notation, each vector $|\mathcal{M}\rangle$ is a colour-singlet state, so colour conservation is simply

$$\left(\sum_j \boldsymbol{T}_j\right)\,|\mathcal{M}\rangle = 0\,, \tag{2.7}$$

where the sum over j extends over all the external partons of the state vector $|\mathcal{M}\rangle$, and the equation is valid order by order in the loop expansion of eq. (2.3).

JHEP06(2005)024

The colour-charge algebra for $\sum_n T_i^n T_k^n \equiv \boldsymbol{T}_i \cdot \boldsymbol{T}_k$ is

$$\boldsymbol{T}_i \cdot \boldsymbol{T}_k = \boldsymbol{T}_k \cdot \boldsymbol{T}_i \qquad \text{if} \qquad i \neq k; \qquad \boldsymbol{T}_i^2 = C_i \,. \tag{2.8}$$

Here C_i is the quadratic Casimir operator in the representation of particle i and we have $C_{\mathrm{F}} = T_{\mathrm{R}}(N_{\mathrm{c}}^2-1)/N_{\mathrm{c}} = (N_{\mathrm{c}}^2-1)/(2N_{\mathrm{c}})$ in the fundamental and $C_{\mathrm{A}} = 2\,T_{\mathrm{R}} N_{\mathrm{c}} = N_{\mathrm{c}}$ in the adjoint representation, i.e. we are using the customary normalization $T_{\mathrm{R}} = 1/2$.

3. Factorization in the singly-unresolved collinear and soft limits

As explained in the introduction in order to devise the approximate matrix elements, we have to study the factorization properties of the relevant squared matrix elements when one, or two partons become soft or collinear to another parton. The factorization of the tree-level matrix elements when one parton becomes soft or collinear to another one are well known [35, 36]. In order to introduce a new notation, in this section we recall the relevant formulae for a tree-level matrix element $|\mathcal{M}_{m+2}^{(0)}\rangle$ with $m+2$ massless QCD partons in the final state and two massless and colourless particles in the initial state.

3.1 The collinear limit

We define the collinear limit of two final-state momenta p_i and p_r with the help of an auxiliary light-like vector n_{ir}^μ ($n_{ir}^2 = 0$) in the usual way,

$$p_i^\mu = z_i p_{ir}^\mu - k_{r\perp}^\mu - \frac{k_{r\perp}^2}{z_i} \frac{n_{ir}^\mu}{2 p_{ir} n_{ir}} \,, \qquad p_r^\mu = z_r p_{ir}^\mu + k_{r\perp}^\mu - \frac{k_{r\perp}^2}{z_r} \frac{n_{ir}^\mu}{2 p_{ir} n_{ir}} \,, \tag{3.1}$$

where p_{ir}^μ is a light-like momentum that points towards the collinear direction and $k_{r\perp}$ is the momentum component that is orthogonal to both p_{ir} and n_{ir} ($p_{ir} \cdot k_{r\perp} = n_{ir} \cdot k_{r\perp} = 0$). Momentum conservation requires that $z_i + z_r = 1$. The two-particle invariant masses ($s_{ir} \equiv 2 p_i p_r$) of the final-state partons are

$$s_{ir} = -\frac{k_{r\perp}^2}{z_i z_r} \,. \tag{3.2}$$

The collinear limit is defined by the uniform rescaling

$$k_{r\perp} \to \lambda k_{r\perp} \,, \tag{3.3}$$

and taking the limit $\lambda \to 0$, when the squared matrix element of an $(m+2)$-parton process has the following asymptotic form,

$$\left| \mathcal{M}_{m+2}^{(0)}(p_i, p_r, \ldots) \right|^2 \simeq 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \frac{1}{s_{ir}} \langle \mathcal{M}_{m+1}^{(0)}(p_{ir}, \ldots) | \hat{P}_{f_i f_r}^{(0)}(z_i, z_r, k_{r\perp}; \varepsilon) | \mathcal{M}_{m+1}^{(0)}(p_{ir}, \ldots) \rangle \,. \tag{3.4}$$

In eq. (3.4) the meaning of the $\simeq$ sign is that we have neglected subleading terms (in this case those that are less singular than $1/\lambda^2$). The $(m+1)$-parton matrix elements on the right-hand side of eq. (3.4) are obtained from the $(m+2)$-parton matrix elements by removing partons i and r and replacing them with a single parton denoted by ir.

JHEP06(2005)024

The parton ir carries the quantum numbers of the pair $i+r$ in the collinear limit: its momentum is p_{ir}^{μ} and its other quantum numbers (flavour, colour) are obtained according to the following rule: anything + gluon gives anything and quark + antiquark gives gluon. The kernels $\hat{P}_{f_i f_r}^{(0)}$ are the d-dimensional Altarelli-Parisi splitting functions, which depend on the momentum fractions of the decay products and on the relative transverse momentum of the pair. For the sake of simplicity, we label the momentum fraction belonging to a certain parton flavour with the corresponding label of the squared matrix element, $z_{f_i} = z_i$. In the case of splitting into a pair, only one momentum fraction is independent, yet, we find it more convenient to keep the functional dependence on both z_i and z_r. Depending on the f_i flavours of the splitting products the explicit functional forms are

$$\langle\mu|\hat{P}_{g_i g_r}^{(0)}(z_i, z_r, k_\perp; \varepsilon)|\nu\rangle = 2C_{\mathrm{A}}\left[-g^{\mu\nu}\left(\frac{z_i}{z_r}+\frac{z_r}{z_i}\right) - 2(1-\varepsilon)z_i z_r \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right], \tag{3.5}$$

$$\langle\mu|\hat{P}_{\bar{q}_i q_r}^{(0)}(z_i, z_r, k_\perp; \varepsilon)|\nu\rangle = T_{\mathrm{R}}\left[-g^{\mu\nu} + 4 z_i z_r \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right], \tag{3.6}$$

$$\langle r|\hat{P}_{q_i g_r}^{(0)}(z_i, z_r; \varepsilon)|s\rangle = \delta_{rs} C_{\mathrm{F}}\left[\frac{1+z_i^2}{z_r} - \varepsilon z_r\right] \equiv \delta_{rs} P_{q_i g_r}^{(0)}(z_i, z_r; \varepsilon), \tag{3.7}$$

where in the last equation we introduced our notation for the spin-averaged splitting function,

$$P_{f_i f_r}(z_i, z_r; \varepsilon) \equiv \langle \hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \varepsilon)\rangle . \tag{3.8}$$

The gluon-gluon and quark-antiquark splittings are symmetric in the momentum fractions of the two decay products, while the quark-gluon splitting is not. Nevertheless, we do not distinguish the flavour kernels $\hat{P}_{qg}$ and $\hat{P}_{gq}$. The ordering of the flavour indices and arguments of the Altarelli-Parisi kernels has no meaning in our notation, i.e.,

$$\hat{P}_{f_i f_r}(z_i, z_r; \varepsilon) = \hat{P}_{f_r f_i}(z_r, z_i; \varepsilon) . \tag{3.9}$$

Thus, it is sufficient to record the kernel belonging to one ordering. We keep this convention throughout.

In order to simplify further discussion, we introduce a symbolic operator $\mathbf{C}_{ir}$ that performs the action of taking the collinear limit of the squared matrix element, keeping the leading singular term. Thus we can write eq. (3.4) as

$$\mathbf{C}_{ir}\left|\mathcal{M}_{m+2}^{(0)}(p_i, p_r, \ldots)\right|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\langle\mathcal{M}_{m+1}^{(0)}(p_{ir}, \ldots)|\hat{P}_{f_i f_r}^{(0)}|\mathcal{M}_{m+1}^{(0)}(p_{ir}, \ldots)\rangle . \tag{3.10}$$

3.2 The soft limit

The soft limit is defined by parametrizing the soft momentum as $p_r = \lambda q_r$ and letting $\lambda \to 0$ at fixed q_r. Neglecting terms that are less singular than $1/\lambda^2$, the soft limit of the squared matrix element can be written as

$$\left|\mathcal{M}_{m+2}^{(0)}(p_r, \ldots)\right|^2 \simeq -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_{i,k}\frac{1}{2}\mathcal{S}_{ik}(r)|\mathcal{M}_{m+1;(i,k)}^{(0)}(\ldots)|^2 , \tag{3.11}$$

JHEP06(2005)024

where

$$\mathcal{S}_{ik}(r) = \frac{2s_{ik}}{s_{ir}s_{rk}} \tag{3.12}$$

is the eikonal factor. In eq. (3.11) the $(m+1)$-parton matrix element on the right-hand side is obtained from the $(m+2)$-parton matrix element on the left-hand side by simply removing the soft parton.

Similarly to the $\mathbf{C}_{ir}$ operator, we introduce the symbolic operator $\mathbf{S}_r$ that performs the action of taking the soft limit of the squared matrix element, keeping the leading singular terms. Thus we can write eq. (3.11) as

$$\mathbf{S}_r \left|\mathcal{M}^{(0)}_{m+2}(p_r,\ldots)\right|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \sum_{i,k} \frac{1}{2}\mathcal{S}_{ik}(r)|\mathcal{M}^{(0)}_{m+1;(i,k)}(\ldots)|^2\,, \tag{3.13}$$

if r is a gluon and $\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}(p_r,\ldots)|^2 = 0$ if r is a quark.

3.3 Matching the singly-unresolved limits

If we want to regularize the squared matrix elements in all singly-unresolved regions of the phase space then we have to subtract all possible collinear and soft limits, i.e. subtract the sum

$$\sum_r \left(\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r\right)\left|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\ldots)\right|^2\,, \tag{3.14}$$

where the 1/2 symmetry factor takes into account that in the summation each collinear configuration is taken into account twice. Subtracting eq. (3.14) we perform a double subtraction in some regions of the phase space where the soft and collinear limits overlap. The collinear limit of eq. (3.13) when the soft gluon r becomes simultaneously collinear to parton i is

$$\mathbf{C}_{ir}\mathbf{S}_r \left|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\ldots)\right|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{2}{s_{ir}}\sum_{k\neq i}\frac{z_i}{z_r}\left|\mathcal{M}^{(0)}_{m+1;(i,k)}(p_i,\ldots)\right|^2\,. \tag{3.15}$$

The factor z_i/z_r is independent of k, therefore, using colour conservation (eq. (2.7)) we can perform the summation and obtain,

$$\mathbf{C}_{ir}\mathbf{S}_r \left|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\ldots)\right|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{2}{s_{ir}}\frac{z_i}{z_r}\boldsymbol{T}_i^2\left|\mathcal{M}^{(0)}_{m+1}(p_i,\ldots)\right|^2\,. \tag{3.16}$$

Similarly, the soft limit of eq. (3.10) when r is a gluon and $z_r \to 0$ is

$$\mathbf{S}_r\mathbf{C}_{ir} \left|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\ldots)\right|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\boldsymbol{T}_i^2\frac{2}{z_r}\left|\mathcal{M}^{(0)}_{m+1}(p_i,\ldots)\right|^2\,. \tag{3.17}$$

Eqs. (3.16) and (3.17) differ by the factor $z_i = 1 - z_r$ in the numerator of eq. (3.16), in which z_r becomes subleading if r is soft. Therefore, eq. (3.16) can be used to account for the double subtraction: it cancels the soft subtraction in the collinear limit by construction,

$$\mathbf{C}_{ir}(\mathbf{S}_r - \mathbf{C}_{ir}\mathbf{S}_r)\left|\mathcal{M}^{(0)}_{m+2}\right|^2 = 0\,, \tag{3.18}$$

JHEP06(2005)024

and the $\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r$ difference is subleading in the soft limit,

$$\mathbf{S}_r(\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r)\left|\mathcal{M}^{(0)}_{m+2}\right|^2 = 0\,. \tag{3.19}$$

Accordingly, in order to remove the double subtraction from eq. (3.14), we have to add terms like that in eq. (3.16). That amounts to always take the collinear limit of the soft factorization formula rather than the reverse (like terms in eq. (3.17)). Thus the candidate for a subtraction term for regularizing the squared matrix element in all singly-unresolved limits is

$$\begin{aligned}\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 &= \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \left(\mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right)\right]|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\ldots)|^2\\ &= 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_r\sum_{i\neq r}\left\{\frac{1}{2}\,\frac{1}{s_{ir}}\,\langle\mathcal{M}^{(0)}_{m+1}(p_{ir},\ldots)|\hat{P}^{(0)}_{f_i f_r}|\mathcal{M}^{(0)}_{m+1}(p_{ir},\ldots)\rangle\, -\right.\\ &\qquad\left. -\,\delta_{f_r g}\sum_{k\neq i,r}\left[\frac{s_{ik}}{s_{ir}s_{rk}} - \frac{2}{s_{ir}}\frac{z_i}{z_r}\right]|\mathcal{M}^{(0)}_{m+1;(i,k)}(\ldots)|^2\right\}.\end{aligned} \tag{3.20}$$

Note that the cancellation of the collinear terms in the soft limit actually requires the symmetry factor multiplying the collinear term, but not the collinear-soft one. This feature will be valid throughout the paper.

The formula in eq. (3.20) cannot be used as a true subtraction term. The reason is that the factorization formulae are valid in the strict collinear and/or soft limits. In order to define subtraction formulae over a finite part of the phase space, we either have to specify the momentum that becomes unresolved (general solutions are presented in refs. [3, 4]), or exact momentum conservation has to be implemented (as in ref. [6]). In the latter case in addition to the poles that are shown explicitly, these subtraction terms contain singularities in the $(m+1)$-parton matrix elements,

$$\mathbf{C}_{js}\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \mathbf{C}_{js}|\mathcal{M}^{(0)}_{m+2}|^2 + (\mathbf{C}_{js}\mathbf{A}_1 - \mathbf{C}_{js})\,|\mathcal{M}^{(0)}_{m+2}|^2\,, \tag{3.21}$$

$$\mathbf{S}_s\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \mathbf{S}_s|\mathcal{M}^{(0)}_{m+2}|^2 + (\mathbf{S}_s\mathbf{A}_1 - \mathbf{S}_s)\,|\mathcal{M}^{(0)}_{m+2}|^2\,, \tag{3.22}$$

where

$$\begin{aligned}&(\mathbf{C}_{js}\mathbf{A}_1 - \mathbf{C}_{js})\,|\mathcal{M}^{(0)}_{m+2}|^2 =\\ &\quad= \sum_{r\neq j,s}\mathbf{C}_{js}\left(\mathbf{S}_r - \mathbf{C}_{sr}\mathbf{S}_r - \mathbf{C}_{jr}\mathbf{S}_r + \sum_{i\neq r,j,s}\left(\frac{1}{2}\,\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r\right)\right)|\mathcal{M}^{(0)}_{m+2}|^2\,,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(\mathbf{S}_s\mathbf{A}_1 - \mathbf{S}_s)\,|\mathcal{M}^{(0)}_{m+2}|^2 =\\ &\quad= \sum_{r\neq s}\mathbf{S}_s\left(\mathbf{S}_r - \mathbf{C}_{sr}\mathbf{S}_r + \sum_{i\neq r,s}\left(\frac{1}{2}\,\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r\right)\right)|\mathcal{M}^{(0)}_{m+2}|^2\,.\end{aligned} \tag{3.24}$$

The terms in eqs. (3.23) and (3.24) are the iterated singly-unresolved limits that we shall define precisely in section 6. In a NLO calculation these terms give singularities in the doubly-unresolved parts of the phase space, where the measurement function, that defines the physical quantity (see section 9), becomes zero, and thus screens the divergencies. The same is not true in a NNLO calculation, which we shall tackle in section 8.

JHEP06(2005)024

4. Doubly-unresolved limits

In this section we discuss all the various limits when partons r and s are simultaneously unresolved, which are needed for constructing an approximate cross section for regularizing the squared matrix element in the doubly-unresolved regions of the phase space. The infrared factorization in the various doubly-unresolved regions was discussed in ref. [37], but without aiming at combining the different factorization formulae such that double counting among the various expressions relevant for single and double soft and/or collinear limits is avoided. In order to set our notation and also exhibit these common terms, we rewrite the published formulae.

At NNLO, the dependence of the squared matrix element $|\mathcal{M}_{m+2}|^2$ on the momenta of the final-state partons after integration over the phase space leads to leading singularities in five different situations,

1. the emission of a collinear parton-triplet;
2. the emission of two pairs of collinear partons;
3. the emission of a soft gluon and a pair of collinear partons;
4. the emission of a soft quark-antiquark pair;
5. the emission of two soft gluons.

The first case is the triply-collinear one, when three final state partons become collinear. The next one corresponds to the doubly-collinear region, where two final-state partons become collinear to two other final-state partons, but not to one another. The third case occurs in the soft-collinear region, where the momentum of a gluon becomes soft in any fixed direction and, at the same time, two partons become collinear. The last two situations occur in the soft region, where either the momenta p_r and p_s of a final-state quark-antiquark pair tend to zero, with the directions of p_r and p_s as well as p_r/p_s fixed, or the momenta of two final-state gluons tend to zero in any fixed direction. In the following we recall the most singular behaviour of the squared matrix element in those regions of phase space.

4.1 Emission of a collinear-parton triplet

We consider three final-state partons i, r and s that can be produced in the splitting process $irs \to i + r + s$. The partons i, r and s have momenta p_i, p_r and p_s. We introduce the following Sudakov parametrization of the parton momenta,

$$p_j^\mu = z_j p_{irs}^\mu + k_{j\perp}^\mu - \frac{k_{j\perp}^2}{z_j} \frac{n_{irs}^\mu}{2p_{irs}n_{irs}}\,, \qquad j = i, r, s\,, \tag{4.1}$$

where p_{irs}^μ is a light-like momentum that points towards the collinear direction, n_{irs}^μ is an auxiliary light-like vector ($n_{irs}^2 = 0$) and $k_{j\perp}$ are the momentum components that are orthogonal to both p_{irs} and n_{irs} ($p_{irs} \cdot k_{j\perp} = n_{irs} \cdot k_{j\perp} = 0$). The variables z_j and $k_{j\perp}$ satisfy the constraints $z_i + z_r + z_s = 1$ and $k_{i\perp} + k_{r\perp} + k_{s\perp} = 0$.

JHEP06(2005)024

The two-particle invariant masses of the final-state partons are

$$(p_j + p_l)^2 = -z_j z_l \left(\frac{k_{j\perp}}{z_j} - \frac{k_{l\perp}}{z_l}\right)^2 , \qquad j,l = i,r,s\,. \tag{4.2}$$

The triply-collinear region is identified by performing the uniform rescaling

$$k_{j\perp} \to \lambda k_{j\perp}\,, \quad j = i,r,s\,, \tag{4.3}$$

and studying the limit $\lambda \to 0$. Again, we introduce the symbolic operator $\mathbf{C}_{irs}$ that keeps the leading singular ($\mathrm{O}(1/\lambda^4)$) terms of the squared matrix element in this limit. According to ref. [37], the squared matrix element $|\mathcal{M}_{m+2}|^2$ behaves as,

$$\begin{aligned}
&\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = (8\pi\alpha_\mathrm{s}\mu^{2\varepsilon})^2 \langle\mathcal{M}^{(0)}_{m}(p_{irs},\ldots)|\,\frac{1}{s_{irs}^2}\hat{P}^{(0)}_{f_i f_r f_s}(\{z_j,s_{jl},k_{j\perp}\};\varepsilon)\,|\mathcal{M}^{(0)}_{m}(p_{irs},\ldots)\rangle\,,
\end{aligned} \tag{4.4}$$

where $s_{irs} = (p_i + p_r + p_s)^2$ and $|\mathcal{M}^{(0)}_{m}(p_{irs},\ldots)\rangle$ corresponds to the m-parton matrix element that is obtained from the $(m+2)$-parton matrix element by replacing the three partons i,r,s by the single parton irs[1].

In general the kernel $\hat{P}_{f_i f_r f_s}(\{z_j,s_{jl},k_{j\perp}\};\varepsilon)$ in eq. (4.4) contains spin correlations of the parent parton. However, in the case of splitting processes that involve a fermion as a parent parton, the spin correlations are absent, therefore, we can write the corresponding spin-dependent splitting function in terms of its average over the polarizations of the parent fermion,

$$\langle s|\hat{P}_{q f_r f_s}(\{z_j,s_{jl},k_{j\perp}\};\varepsilon)|s'\rangle = \delta_{ss'} P_{q f_r f_s}(\{z_j,s_{jl}\};\varepsilon)\,, \tag{4.5}$$

where we used our generic notation for the spin-averaged splitting function (cf. eq. (3.8)). The spin-averaged splitting function is presented in refs. [31, 37] explicitly. In order to better exhibit the soft structure of these kernels, we give different forms of the same functions. For the $q \to q\bar{q}'q'$ process we write

$$\begin{aligned}
&\frac{1}{s_{irs}^2} P_{q_i\bar{q}'_r q'_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs};\varepsilon) = \\
&\quad = C_\mathrm{F} T_\mathrm{R}\Bigg\{\frac{1}{s_{irs}s_{rs}}\Bigg[\frac{z_i}{z_r+z_s} - \frac{s_{ir}z_s + s_{is}z_r}{s_{rs}(z_r+z_s)} + \frac{s_{ir}s_{is}}{s_{rs}s_{irs}} + \frac{s_{irs}}{s_{rs}}\frac{z_r z_s}{(z_r+z_s)^2} - \\
&\qquad\qquad - \frac{z_r z_s}{z_r+z_s} + \frac{1-\varepsilon}{2}\left(z_r + z_s - \frac{s_{rs}}{s_{irs}}\right)\Bigg] + (r \leftrightarrow s)\Bigg\}\,.
\end{aligned} \tag{4.6}$$

We decompose the splitting functions that involve one, or two gluons into abelian and non-abelian contributions. For the $q \to qgg$ splitting we only need the spin-averaged functions,

$$P_{qgg} = C_\mathrm{F}\, P^{(\mathrm{ab})}_{qgg} + C_\mathrm{A}\, P^{(\mathrm{nab})}_{qgg}\,, \tag{4.7}$$

[1] According to the convention of eq. (3.9), we have Altarelli-Parisi kernels for all the permutations of the flavour indices, which are obtained by permuting suitably the respective momenta in their arguments; e.g. P_{gqg} is obtained from P_{qgg} by swapping the first two momenta.

JHEP06(2005)024

where for the abelian part we write

$$\frac{1}{s_{irs}^2} P^{(\mathrm{ab})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) =$$
$$= C_{\mathrm{F}} \Bigg\{ \left(\frac{1-z_s}{s_{irs} s_{ir}} + \frac{1-z_r}{s_{irs} s_{is}} + \frac{z_i}{s_{ir} s_{is}} \right) \frac{1}{z_r} \left(\frac{1+z_i^2}{1-z_i} - \varepsilon z_s - \varepsilon (1+\varepsilon) \frac{z_r}{2} \right) +$$
$$+ \frac{1-\varepsilon}{s_{irs}^2} \left[\varepsilon - \frac{s_{irs}}{s_{ir}} (1+\varepsilon)(3 - z_r - 2 z_s) - \frac{s_{is}}{s_{ir}} (1-\varepsilon) \right] + (r \leftrightarrow s) \Bigg\}, \quad (4.8)$$

and for the non-abelian part we use

$$\frac{1}{s_{irs}^2} P^{(\mathrm{nab})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) =$$
$$= C_{\mathrm{F}} \Bigg\{ \frac{1}{s_{irs} s_{rs}} \Bigg[(1-\varepsilon) \left(\frac{s_{ir} z_s + s_{is} z_r}{s_{rs}(z_r + z_s)} - \frac{s_{ir} s_{is}}{s_{irs} s_{rs}} - \frac{s_{irs}}{s_{rs}} \frac{z_r z_s}{(z_r + z_s)^2} \right) -$$
$$- z_i \left(\frac{4}{z_r + z_s} - \frac{1}{z_r} \right) \Bigg] - \frac{1}{s_{irs} s_{ir}} \frac{(1-z_r)^2 + (1-z_s)^2}{2 z_r (z_r + z_s)} -$$
$$- \frac{1}{s_{ir} s_{is}} \frac{z_i}{2 z_r} \frac{1+z_i^2}{z_r + z_s} + \frac{1}{2 s_{ir} s_{rs}} \left[\frac{1+z_i^2}{z_s} + \frac{1+(1-z_s)^2}{z_r + z_s} \right] +$$
$$+ \frac{(1-\varepsilon)^2}{2 s_{irs}^2} + \frac{1}{s_{irs} s_{rs}} \left[(1-\varepsilon) \left(\frac{2}{z_r} - \frac{1}{z_r + z_s} \right) \frac{(z_r - z_s)^2}{4} - 1 \right] +$$
$$+ \frac{1}{2 s_{irs} s_{ir}} \left[\frac{1+(1-z_s)^2}{z_r} - \frac{4 - 2 z_s + z_s^2 - z_r}{z_r + z_s} \right] +$$
$$+ \frac{\varepsilon}{2} \Bigg[\frac{1}{s_{irs} s_{ir}} \left((1-z_s) \left(\frac{z_r}{z_s} + \frac{z_s}{z_r} - \varepsilon \right) - \frac{z_r^2 (1-z_s)}{z_s (z_r + z_s)} \right) + \frac{z_i}{s_{ir} s_{is}} \times$$
$$\times \left(\frac{z_r}{z_s} + \frac{1+\varepsilon}{2} \right) - \frac{1}{s_{ir} s_{rs}} \left(\frac{(z_r + z_s)^2}{z_s} + \frac{z_s^2}{z_r + z_s} \right) \Bigg] + (r \leftrightarrow s) \Bigg\}. \quad (4.9)$$

In the case of the $g \to g q \bar{q}$ process we decompose the spin-dependent splitting function as

$$\langle \mu | \hat{P}_{g q \bar{q}} | \nu \rangle = C_{\mathrm{F}} \, \langle \mu | \hat{P}^{(\mathrm{ab})}_{g q \bar{q}} | \nu \rangle + C_{\mathrm{A}} \, \langle \mu | \hat{P}^{(\mathrm{nab})}_{g q \bar{q}} | \nu \rangle \, . \quad (4.10)$$

The explicit form of the abelian term is

$$\frac{1}{s_{irs}^2} \langle \mu | \hat{P}^{(\mathrm{ab})}_{g_i q_r \bar{q}_s}(s_{ir}, s_{is}, s_{rs}, k_{i\perp}, k_{r\perp}, k_{s\perp}) | \nu \rangle =$$
$$= T_{\mathrm{R}} \Bigg\{ - g^{\mu\nu} \left[\frac{1}{s_{ir} s_{is}} - \frac{1}{s_{irs}^2} - \frac{2}{s_{irs} s_{ir}} \left(1 - \frac{1-\varepsilon}{2} \frac{s_{ir} + s_{is}}{s_{irs}} \right) \right] +$$
$$+ \frac{2}{s_{irs} s_{ir} s_{is}} \left[k^\mu_{r\perp} k^\nu_{s\perp} + k^\nu_{r\perp} k^\mu_{s\perp} - (1-\varepsilon) k^\mu_{i\perp} k^\nu_{i\perp} \right] + (r \leftrightarrow s) \Bigg\}, \quad (4.11)$$

and that of the non-abelian term is

$$\frac{1}{s_{irs}^2} \langle \mu | \hat{P}^{(\mathrm{nab})}_{g_i q_r \bar{q}_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{i\perp}, k_{r\perp}, k_{s\perp}) | \nu \rangle =$$
$$= T_{\mathrm{R}} \Bigg\{ - \frac{g^{\mu\nu}}{s_{irs} s_{rs}} \left[\frac{z_i}{z_r + z_s} - \frac{s_{ir} z_s + s_{is} z_r}{s_{rs}(z_r + z_s)} + \frac{s_{ir} s_{is}}{s_{rs} s_{irs}} + \frac{s_{irs}}{s_{rs}} \frac{z_r z_s}{(z_r + z_s)^2} \right] +$$

JHEP06(2005)024

$$
\begin{aligned}
&+\frac{1}{s_{irs}s_{rs}}\frac{1}{z_i}\left(-g^{\mu\nu}+\frac{4z_rz_s}{z_r+z_s}\frac{k^\mu_{rs\perp}k^\nu_{rs\perp}}{k^2_{rs\perp}}\right)-\frac{g^{\mu\nu}}{2z_i}\left(\frac{z_r}{s_{ir}s_{rs}}-\frac{1}{s_{irs}s_{rs}}\right)-\\
&-g^{\mu\nu}\left[\frac{1-z_r}{2s_{irs}s_{is}}\left(\frac{1}{z_i}+\frac{1}{z_r+z_s}\right)-\frac{1}{2s_{ir}s_{is}}-\frac{1-\varepsilon}{2s^2_{irs}}-\frac{z_r}{2s_{ir}s_{rs}}\frac{1}{z_r+z_s}\right]+\\
&+\frac{1}{s_{irs}s_{ir}s_{rs}}\left[2(1-\varepsilon)k^\mu_{r\perp}k^\nu_{r\perp}-\frac{4z_r^2}{z_i(z_r+z_s)}k^\mu_{s\perp}k^\nu_{s\perp}+\right.\\
&\qquad\left.+\left(\frac{2z_r(z_s-z_i)}{z_i(z_r+z_s)}+1-\varepsilon\right)\left(k^\mu_{r\perp}k^\nu_{s\perp}+k^\mu_{s\perp}k^\nu_{r\perp}\right)\right]-\\
&\left.-\frac{1}{s_{irs}s_{ir}s_{is}}\left[k^\mu_{r\perp}k^\nu_{s\perp}+k^\nu_{r\perp}k^\mu_{s\perp}-(1-\varepsilon)k^\mu_{i\perp}k^\nu_{i\perp}\right]+(r\leftrightarrow s)\right\}, \qquad (4.12)
\end{aligned}
$$

where we have introduced the abbreviation

$$
k^\mu_{rs\perp}\equiv\frac{k^\mu_{r\perp}}{z_r}-\frac{k^\mu_{s\perp}}{z_s}\,. \qquad (4.13)
$$

The pure-gluon splitting kernel as given in eq. (66) of ref. [37] is completely symmetric in its indices. Therefore, without loss of definiteness, we can refer to that formula and obtain our definition from that by making the change of indices $1\to i$, $2\to r$ and $3\to s$ and then the formal substitution $\tilde{k}_{j\perp}\to k_{j\perp}$.

The triply-collinear factorization formulae of this section are valid in the strict collinear limits. If these expressions are to be used as subtraction terms, the momentum fractions z_j ($j=i,\,r,\,s$) have to be defined over the extended phase space region where the subtraction is defined. This definition has to be such that z_j vanishes only if the corresponding parton j becomes soft. Otherwise, in the kernels $\hat{P}_{qgg}$, $\hat{P}^{(\mathrm{nab})}_{gq\bar{q}}$ and $\hat{P}_{ggg}$ we introduce spurious poles in those terms that contain $1/z_j$ factors. The energy fractions $z_j=E_j/(E_i+E_r+E_s)$ fulfill such a requirement over the whole phase space. Other definitions, for instance, in terms of invariants, like $z_j=p_j\cdot n/[(p_i+p_r+p_s)\cdot n]$, with n being a light-like momentum, e.g. the momentum of a hard parton k, lead to spurious poles unless the region where the momentum p_j is collinear to the momentum n is excluded, e.g. if the subtraction term is defined there to be zero.

4.2 Emission of two collinear pairs of partons

The double collinear limit occurs when two pairs of collinear partons are emitted. The limit is precisely defined by the usual Sudakov parametrization eq. (3.1) for the two pairs separately. We consider two pairs of final-state partons i, r and j, s that can be produced in the splitting processes $ir\to i+r$ and $js\to j+s$. The partons i and r have momenta p_i, p_r, their parametrization is given in eq. (3.1). The parametrization of the second splitting is completely analogous in terms of momenta p^μ_{js}, $k^\mu_{s\perp}$, n^μ_{js} and parameter z_j.

The doubly-collinear region is identified by performing the uniform rescaling

$$
k_{r\perp}\to\lambda k_{r\perp}\,,\qquad k_{s\perp}\to\lambda k_{s\perp}\,, \qquad (4.14)
$$

JHEP06(2005)024

and studying the limit $\lambda \to 0$. Neglecting terms that are less singular than $1/\lambda^4$, the squared matrix element $|\mathcal{M}_{m+2}|^2$ behaves as [37]

$$\mathbf{C}_{ir;js} \left|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_j, p_s, \ldots)\right|^2 = (8\pi\alpha_s\mu^{2\varepsilon})^2 \frac{1}{s_{ir}s_{js}} \times \\ \times \langle \mathcal{M}^{(0)}_m(p_{ir}, p_{js}, \ldots)|\hat{P}^{(0)}_{f_i f_r}(z_i, z_r, k_{r\perp}; \varepsilon)\hat{P}^{(0)}_{f_j f_s}(z_j, z_s, k_{s\perp}; \varepsilon)|\mathcal{M}^{(0)}_m(p_{ir}, p_{js}, \ldots)\rangle\,, \tag{4.15}$$

where $|\mathcal{M}^{(0)}_m(p_{ir}, p_{js}, \ldots)\rangle$ corresponds to the m-parton matrix element that is obtained from the $(m+2)$-parton matrix element by replacing the two partons i and r by the single parton ir and the other two partons j, s by the single parton js. The d-dimensional Altarelli-Parisi splitting functions were given in eqs. (3.5)–(3.7). We see from eq. (4.15) that the doubly-collinear limit factorizes completely as

$$\mathbf{C}_{ir;js}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_j, p_s, \ldots)|^2 = \mathbf{C}_{ir}\mathbf{C}_{js}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_j, p_s, \ldots)|^2\,. \tag{4.16}$$

4.3 Soft-collinear limit

Next we consider the tree-level matrix element of $m + 2$ partons in the limit where the momentum p_s of a final-state gluon becomes soft and the momenta p_i and p_r of two *other* final-state partons become collinear. The soft limit is defined by the rescaling $p_s = \lambda_s q_s$ with letting $\lambda_s \to 0$. The collinear limit is defined by the usual Sudakov parametrization of eq. (3.1). The limits are taken uniformly, $\lambda_r = \lambda_s \equiv \lambda$. In this limit, the squared matrix element $|\mathcal{M}_{m+2}|^2$ behaves as [37]

$$\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_s, \ldots)|^2 = \\ = -(8\pi\alpha_s\mu^{2\varepsilon})^2 \frac{1}{s_{ir}} \left[\sum_j \sum_{l\neq j} \frac{1}{2}\mathcal{S}_{jl}(s) \times \right. \\ \left. \times \langle \mathcal{M}^{(0)}_m(p_{ir}, \ldots)|\, \boldsymbol{T}_j \cdot \boldsymbol{T}_l\, \hat{P}_{f_i f_r}(z_i, z_r, k_{r\perp}; \varepsilon)|\mathcal{M}^{(0)}_m(p_{ir}, \ldots)\rangle\right], \tag{4.17}$$

where $|\mathcal{M}^{(0)}_m(p_{ir}, \ldots)\rangle$ corresponds to the m-parton matrix element that is obtained from the $(m+2)$-parton matrix element by omitting the gluon s and replacing the two partons i, r by the single parton ir. In the terms of eq. (4.17), when either j or l is equal to (ir), the colour charge operator is $\boldsymbol{T}_{ir} = \boldsymbol{T}_i + \boldsymbol{T}_r$ and

$$\mathcal{S}_{j(ir)}(s) = \frac{2s_{j(ir)}}{s_{js}s_{s(ir)}} \equiv \frac{2(s_{ji} + s_{jr})}{s_{js}(s_{si} + s_{sr})}\,. \tag{4.18}$$

In the strict collinear limit

$$\mathcal{S}_{j(ir)}(s) = \mathcal{S}_{ji}(s) = \mathcal{S}_{jr}(s)\,, \tag{4.19}$$

which we shall use later. Of course, $\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_s, \ldots)|^2 = 0$ if parton s is a quark.

4.4 Emission of a soft $\bar{q}q$-pair

In the region of soft $\bar{q}q$-emission the momenta of the quark p_r and that of the antiquark p_s tends to zero such that $p_r = \lambda q_r$ and $p_s = \lambda q_s$ with $\lambda \to 0$ for fixed q_r and q_s. In this limit, the squared matrix element diverges as $1/\lambda^4$ and its most divergent part can be computed in terms of the eikonal current of a soft gluon of momentum $p_r + p_s$ [37],

$$\mathbf{S}_{rs}^{(q\bar{q})}\,|\mathcal{M}_{m+2}(0)(p_r,p_s,\ldots)|^2 = \tag{4.20}$$
$$= (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2 \frac{1}{s_{rs}^2} \sum_{i\neq r,s}\sum_{k\neq i,r,s}\left(\frac{s_{ir}s_{ks}+s_{kr}s_{is}-s_{ik}s_{rs}}{s_{i(rs)}s_{k(rs)}} - 2\frac{s_{ir}s_{is}}{s_{i(rs)}^2}\right) T_{\rm R}\left|\mathcal{M}_{m;(i,k)}^{(0)}(\ldots)\right|^2 ,$$

where the matrix element on the right hand side is obtained from the original $(m+2)$-parton matrix element by omitting the quark-antiquark pair, having momenta p_r and p_s.

4.5 Emission of two soft gluons

We consider the tree-level matrix element of $m+2$ final state partons when a gluon of momentum p_r and another gluon of momentum p_s become simultaneously soft. The limit is precisely defined by rescaling the soft-gluon momenta by an overall factor λ, $p_r = \lambda q_r$ and $p_s = \lambda q_s$, and then performing the limit $\lambda \to 0$ for fixed q_r and q_s. In this limit, the matrix element diverges as $1/\lambda^2$ and its most divergent part can be computed in terms of the two-gluon soft current $J^{\mu_1\mu_2}(q_r,q_s)$, given explicitly in ref. [37].

The singular behaviour of $|\mathcal{M}_{m+2}^{(0)}|^2$ at $\mathrm{O}(1/\lambda^4)$ can be written as a sum of an abelian and a non-abelian term [37],

$$\mathbf{S}_{rs}^{(gg)}|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\ldots)|^2 = \left(\mathbf{S}_{rs}^{(\rm ab)} + \mathbf{S}_{rs}^{(\rm nab)}\right)|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\ldots)|^2 . \tag{4.21}$$

The abelian term contains the product of two one-gluon soft functions, $\mathcal{S}_{ik}(r)$ given in eq. (3.12),

$$\mathbf{S}_{rs}^{(\rm ab)}|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\ldots)|^2 = (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{8}\sum_{i,j,k,l=1}^{m}\mathcal{S}_{ik}(r)\,\mathcal{S}_{jl}(s)\left|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\ldots)\right|^2 . \tag{4.22}$$

The non-abelian contribution is written in terms of $\mathcal{S}_{ik}(r,s)$, the two-gluon soft function introduced in ref. [37],

$$\mathbf{S}_{rs}^{(\rm nab)}|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\ldots)|^2 = -(8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{4}C_{\rm A}\sum_{i,k=1}^{m}\mathcal{S}_{ik}(r,s)\left|\mathcal{M}_{m,(i,k)}^{(0)}(\ldots)\right|^2 . \tag{4.23}$$

The colour-correlated functions $|\mathcal{M}_{m,(i,k)}^{(0)}|^2$ and $|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}|^2$ are given in terms of m-parton amplitudes obtained from the original $(m+2)$-parton amplitude by omitting the gluons r and s (for the definitions see eqs. (2.5) and (2.6), respectively).

It is useful to break the sum in eq. (4.22) into sums according to the number of hard partons in the eikonal factors,

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{m}\mathcal{S}_{ik}(p_r)\mathcal{S}_{jl}(p_s)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}|^2 = \\
&\quad = \sum_i\sum_{j\neq i}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\mathcal{S}_{ik}(r)\mathcal{S}_{jl}(s)\langle\mathcal{M}_m^{(0)}|\,\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\}\,|\mathcal{M}_m^{(0)}\rangle + \\
&\quad\quad + 4\sum_i\sum_{j\neq i}\sum_{k\neq i,j}\mathcal{S}_{ik}(r)\mathcal{S}_{ji}(s)\langle\mathcal{M}_m^{(0)}|\,\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_j\cdot\boldsymbol{T}_i\}\,|\mathcal{M}_m^{(0)}\rangle + \\
&\quad\quad + 2\sum_i\sum_{k\neq i}\mathcal{S}_{ik}(r)\mathcal{S}_{ik}(s)\langle\mathcal{M}_m^{(0)}|\,\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\}\,|\mathcal{M}_m^{(0)}\rangle .
\end{aligned} \tag{4.24}$$

JHEP06(2005)024

The sum in the first line requires at least four hard partons, the next line involves at least three hard partons and the last sum contains at least two hard partons. Therefore, for the case of two hard partons, relevant to two-jet production in electron-positron annihilation, only the last sum contributes.

For later convenience, we also record a new form of the two-gluon soft function. The original expression of ref. [37] in our notation reads

$$\mathcal{S}_{ik}(r,s) = \mathcal{S}_{ik}^{(\mathrm{s.o.})}(r,s) + 4\frac{s_{ir}s_{ks}+s_{is}s_{kr}}{s_{i(rs)}s_{k(rs)}}\left[\frac{1-\varepsilon}{s_{rs}^2} - \frac{1}{8}\mathcal{S}_{ik}^{(\mathrm{s.o.})}(r,s)\right] - \frac{4}{s_{rs}}\mathcal{S}_{ik}(rs)\,, \tag{4.25}$$

where

$$\mathcal{S}_{ik}^{(\mathrm{s.o.})}(r,s) = \mathcal{S}_{ik}(s)\left(\mathcal{S}_{is}(r)+\mathcal{S}_{ks}(r)-\mathcal{S}_{ik}(r)\right) \tag{4.26}$$

is the approximation of the soft function $\mathcal{S}_{ik}(r,s)$ in the strongly-ordered approximation and $\mathcal{S}_{ik}(rs)$ is given by eq. (3.12),

$$\mathcal{S}_{ik}(rs) = \frac{2s_{ik}}{s_{i(rs)}s_{k(rs)}}\,. \tag{4.27}$$

In order to exhibit the relation of this expression to the triply-collinear splitting functions, we rewrite eq. (4.23) using partial fractioning,

$$\begin{aligned}
&\frac{1}{4}\sum_{i,k=1}^{m}\mathcal{S}_{ik}(p_r,p_s)|\mathcal{M}_{m(i,k)}^{(0)}(p_1,\dots p_m)|^2 = \\
&\quad = \frac{1}{s_{rs}}\sum_i\frac{1}{s_{ir}+s_{is}}\sum_{k\neq i}\langle\mathcal{M}_m^{(0)}|\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\,|\mathcal{M}_m^{(0)}\rangle\times\\
&\qquad\times\left[\frac{1}{s_{rs}}(1-\varepsilon)\left(\frac{s_{ir}s_{ks}+s_{kr}s_{is}}{s_{ir}+s_{is}+s_{kr}+s_{ks}} - \frac{s_{ir}s_{is}}{s_{ir}+s_{is}}\right) - \right.\\
&\qquad\quad - s_{ik}\left(\frac{4}{s_{ir}+s_{is}+s_{kr}+s_{ks}} - \frac{1}{s_{is}+s_{kr}}\right) - \\
&\qquad\quad - \frac{s_{rs}s_{ik}^2}{s_{ir}+s_{is}+s_{kr}+s_{ks}}\left(\frac{2}{s_{ir}s_{kr}}+\frac{1}{s_{ir}s_{ks}}\right) + \\
&\qquad\quad + \frac{s_{ik}(s_{ir}+s_{is})}{s_{ir}(s_{ir}+s_{ks})} + \frac{s_{ik}(s_{ir}+s_{is})}{s_{ir}(s_{ir}+s_{is}+s_{kr}+s_{ks})} + \\
&\qquad\quad \left. + \frac{s_{ik}(s_{ir}+s_{is})}{(s_{ir}+s_{ks})(s_{ir}+s_{is}+s_{kr}+s_{ks})}\left(\frac{s_{is}}{s_{ir}} - \frac{s_{is}(s_{kr}+s_{ks})}{s_{ks}(s_{ir}+s_{is})}\right) + (r\leftrightarrow s)\right].
\end{aligned} \tag{4.28}$$

Finally we note that if the two soft partons are of different flavours, the squared matrix element does not have a leading singularity,

$$\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\dots)|^2 = \delta_{f_rf_s}\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}(p_r,p_s,\dots)|^2\,. \tag{4.29}$$

4.6 Matching the doubly-unresolved limits

If we want to regularize the squared matrix elements in all doubly-unresolved regions of the phase space, then we have to subtract all possible collinear and soft limits. Analogously to

JHEP06(2005)024

eq. (3.20), we define the subtraction term $\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$, that must contain the sum

$$\sum_r \sum_{s\neq r}\left[\sum_{i\neq r,s}\left(\frac{1}{6}\,\mathbf{C}_{irs}+\frac{1}{2}\,\mathbf{CS}_{ir;s}+\sum_{j\neq i,r,s}\frac{1}{8}\,\mathbf{C}_{ir;js}\right)+\frac{1}{2}\,\mathbf{S}_{rs}\right]\left|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\ldots)\right|^2 . \tag{4.30}$$

The symmetry factors ensure that, after summation, all collinear configurations are taken into account only once. The various terms in the sum (4.30) overlap in those regions of the phase space where the limits overlap. Therefore, eq. (4.30) contains double and even triple subtractions. We observe that at most triple overlaps may occur because the doubly- and triply-collinear regions do not overlap. Let us first discuss the double subtractions.

The triply-collinear region overlaps with the soft-collinear one. The soft-collinear limit ($z_s \to 0$, $z_r + z_i \to 1$) of the triply-collinear factorization formula, eq. (4.4), is

$$\begin{aligned}&\mathbf{CS}_{ir;s}\mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 =\\ &\quad= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_{ir}^2\,\frac{2}{s_{ir}s_{(ir)s}}\frac{1}{z_s}\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\hat{P}_{f_i f_r}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle\,. \end{aligned}\tag{4.31}$$

In the triply-collinear limit eq. (4.17) behaves as

$$\begin{aligned}&\mathbf{C}_{irs}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 =\\ &\quad= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_{ir}^2\,\frac{2}{s_{ir}s_{(ir)s}}\frac{z_{ir}}{z_s}\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\hat{P}_{f_i f_r}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle\,. \end{aligned}\tag{4.32}$$

The right-hand sides of eqs. (4.31) and (4.32) differ in subleading terms in the soft-collinear limit ($z_{ir} = 1-z_s = 1+\mathrm{O}(\lambda)$). We also record two equivalent forms of eqs. (4.31) and (4.32) that will be useful later. In the $i||r$ collinear limit $s_{is} = z_i s_{(ir)s}$, therefore, eq. (4.31) can also be written as

$$\begin{aligned}&\mathbf{CS}_{ir;s}\mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 =\\ &\quad= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_{ir}^2\,\frac{2}{s_{ir}s_{is}}\frac{z_i}{z_s}\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\hat{P}_{f_i f_r}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle\,. \end{aligned}\tag{4.33}$$

Taking the triply-collinear limit of the soft-collinear formula together with eq. (4.19), we get an identical expression to eq. (4.33),

$$\begin{aligned}&\mathbf{C}_{irs}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 =\\ &\quad= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_{ir}^2\,\frac{2}{s_{ir}s_{is}}\frac{z_i}{z_s}\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\hat{P}_{f_i f_r}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle\,. \end{aligned}\tag{4.34}$$

The fact that eqs. (4.33) and (4.34) are identical while eqs. (4.31) and (4.32) differ explicitly by sub-leading terms is due to the ambiguity in the definition of the soft-collinear limit beyond the leading order, represented by eq. (4.19).

The doubly-collinear region also overlaps with the soft-collinear one. The soft-collinear limit of the doubly-collinear formula, eq. (4.15), is

$$\begin{aligned}&\mathbf{CS}_{ir;s}\mathbf{C}_{ir;js}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\ldots)|^2 =\\ &\quad= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_j^2\,\frac{2}{s_{ir}s_{js}}\frac{1}{z_s}\langle\mathcal{M}_m^{(0)}(p_{ir},p_j,\ldots)|\,\hat{P}_{f_i f_r}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{ir},p_j,\ldots)\rangle\,, \end{aligned}\tag{4.35}$$

JHEP06(2005)024

while in the doubly-collinear limit eq. (4.17) behaves as

$$\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_j,p_s,\ldots)|^2 = \tag{4.36}$$
$$= (8\pi\alpha_s\mu^{2\varepsilon})^2\boldsymbol{T}_j^2\,\frac{2}{s_{ir}s_{js}}\frac{z_j}{z_s}\langle\mathcal{M}^{(0)}_m(p_{ir},p_j,\ldots)|\,\hat{P}^{(0)}_{f_if_r}(z_i,z_r,k_{r\perp};\varepsilon)\,|\mathcal{M}^{(0)}_m(p_{ir},p_j,\ldots)\rangle\,.$$

The difference between eqs. (4.35) and (4.36) is again subleading in the soft-collinear limit ($z_j = 1 - z_s = 1 + \mathrm{O}(\lambda)$).

The region of splitting into a collinear $f_i\bar{q}q$-triplet (parton i can be either a quark or a gluon) overlaps with the soft $q\bar{q}$-emission. In this overlapping region,

$$\mathbf{C}_{irs}\mathbf{S}^{(q\bar{q})}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \tag{4.37}$$
$$= (8\pi\alpha_s\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\,T_\mathrm{R}\frac{2}{s_{i(rs)}s_{rs}}\left[\frac{z_i}{z_r+z_s} - \frac{(s_{ir}z_s - s_{is}z_r)^2}{s_{i(rs)}s_{rs}(z_r+z_s)^2}\right]|\mathcal{M}^{(0)}_m(p_i,\ldots)|^2\,.$$

In the soft-$\bar{q}q$ limit, this formula differs by a subleading term from the doubly-soft limit of the triply-collinear factorization formula,

$$\mathbf{S}^{(q\bar{q})}_{rs}\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \tag{4.38}$$
$$= (8\pi\alpha_s\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\,T_\mathrm{R}\frac{2}{s_{i(rs)}s_{rs}}\left[\frac{1}{z_r+z_s} - \frac{(s_{ir}z_s - s_{is}z_r)^2}{s_{i(rs)}s_{rs}(z_r+z_s)^2}\right]\left|\mathcal{M}^{(0)}_m(p_i,\ldots)\right|^2\,.$$

The soft $q\bar{q}$-emission does not overlap with any other doubly-unresolved regions.

The phase space regions where soft gluon-pair emission occurs overlap with all other doubly-unresolved parts of the phase space (except for the soft-$\bar{q}q$ emission). In the overlapping region of splitting into a collinear f_igg-triplet (parton i can be either a quark or a gluon) and soft gg-emission we obtain

$$\mathbf{C}_{irs}\mathbf{S}^{(gg)}_{rs}|\mathcal{M}^{(0)}_{m+2}|^2 = \mathbf{C}_{irs}\mathbf{S}^{(\mathrm{ab})}_{rs}|\mathcal{M}^{(0)}_{m+2}|^2 + \mathbf{C}_{irs}\mathbf{S}^{(\mathrm{nab})}_{rs}|\mathcal{M}^{(0)}_{m+2}|^2\,, \tag{4.39}$$

where

$$\mathbf{C}_{irs}\mathbf{S}^{(\mathrm{ab})}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = (8\pi\alpha_s\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\boldsymbol{T}_i^2\,\frac{4z_i^2}{s_{ir}s_{is}z_rz_s}|\mathcal{M}^{(0)}_m(p_i,\ldots)|^2\,, \tag{4.40}$$

$$\begin{aligned}\mathbf{C}_{irs}\mathbf{S}^{(\mathrm{nab})}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 &= (8\pi\alpha_s\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\,C_\mathrm{A}\,\times\\ \times\Bigg[\frac{(1-\varepsilon)}{s_{i(rs)}s_{rs}}\frac{(s_{ir}z_s-s_{is}z_r)^2}{s_{i(rs)}s_{rs}(z_r+z_s)^2} &- \frac{z_i}{s_{i(rs)}s_{rs}}\left(\frac{4}{z_r+z_s}-\frac{1}{z_r}\right) -\\ -\frac{1}{s_{i(rs)}s_{ir}}\frac{2z_i^2}{z_r(z_r+z_s)} &- \frac{z_i^2}{s_{i(rs)}s_{is}}\frac{1}{z_r(z_r+z_s)} +\\ +\frac{z_i}{s_{ir}s_{rs}}\left(\frac{1}{z_s}+\frac{1}{z_r+z_s}\right) &+ (r\leftrightarrow s)\Bigg]|\mathcal{M}^{(0)}_m(p_i,\ldots)|^2\,.\end{aligned} \tag{4.41}$$

JHEP06(2005)024

Computing the doubly-soft limits of the $\hat{P}^{(\mathrm{nab})}_{f_i gg}$ splitting kernels,[2] we can derive that

$$\begin{aligned}
&\mathbf{S}^{(gg)}_{rs}\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\dots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\biggl\{\boldsymbol{T}_i^2\,\frac{4}{s_{ir}s_{is}z_r z_s} + \\
&\qquad + C_{\mathrm{A}}\biggl[\frac{(1-\varepsilon)}{s_{i(rs)}s_{rs}}\frac{(s_{ir}z_s - s_{is}z_r)^2}{s_{i(rs)}s_{rs}(z_r+z_s)^2} - \frac{1}{s_{i(rs)}s_{rs}}\left(\frac{4}{z_r+z_s} - \frac{1}{z_r}\right) - \\
&\qquad - \frac{2}{s_{i(rs)}s_{ir}}\frac{1}{z_r(z_r+z_s)} - \frac{1}{s_{i(rs)}s_{is}}\frac{1}{z_r(z_r+z_s)} + \\
&\qquad + \frac{1}{s_{ir}s_{rs}}\left(\frac{1}{z_s}+\frac{1}{z_r+z_s}\right) + (r\leftrightarrow s)\biggr]\biggr\}|\mathcal{M}^{(0)}_{m}(p_i,\dots)|^2\,. \qquad (4.42)
\end{aligned}$$

We see that in the doubly-soft limit eqs. (4.37) and (4.38) differ by subleading terms, as well as the sum of eqs. (4.40) and (4.41) differ from eq. (4.42) by subleading terms ($z_i = 1 - z_r - z_s = 1 + \mathrm{O}(\lambda)$).

In the doubly-collinear limit, the doubly-soft factorization formula simplifies to

$$\begin{aligned}
&\mathbf{C}_{ir;js}\mathbf{S}^{(gg)}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_j,p_s,\dots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\boldsymbol{T}_i^2\,\frac{2}{s_{ir}}\frac{z_i}{z_r}\,\boldsymbol{T}_j^2\,\frac{2}{s_{js}}\frac{z_j}{z_s}\,|\mathcal{M}^{(0)}_{m}(p_i,p_j,\dots)|^2\,. \qquad (4.43)
\end{aligned}$$

In the region of the doubly-soft gluon emission, the limit of the doubly-collinear formula,

$$\begin{aligned}
&\mathbf{S}^{(gg)}_{rs}\mathbf{C}_{ir;js}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_j,p_s,\dots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\boldsymbol{T}_i^2\,\frac{2}{s_{ir}z_r}\,\boldsymbol{T}_j^2\,\frac{2}{s_{js}z_s}\,|\mathcal{M}^{(0)}_{m}(p_i,p_j,\dots)|^2\,, \qquad (4.44)
\end{aligned}$$

differs from eq. (4.43) by subleading terms ($z_i = 1-z_r = 1+\mathrm{O}(\lambda)$, $z_j = 1-z_s = 1+\mathrm{O}(\lambda)$).

The last doubly-overlapping region is the overlap of the soft-collinear region with the emission of two soft gluons. In the soft-collinear limit eq. (4.21) becomes

$$\begin{aligned}
&\mathbf{CS}_{ir;s}\mathbf{S}^{(gg)}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\dots)|^2 = \\
&\quad = -(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{2}{s_{ir}}\,\frac{z_i}{z_r}\boldsymbol{T}_i^2\,\sum_j\sum_{l\neq j}\frac{1}{2}\mathcal{S}_{jl}(s)|\mathcal{M}^{(0)}_{m;(j,l)}(p_i,\dots)|^2\,. \qquad (4.45)
\end{aligned}$$

Note that the non-abelian part of eq. (4.21) does not yield a leading singularity in this case. In the same region, the doubly-soft limit of the soft-collinear formula is

$$\begin{aligned}
&\mathbf{S}^{(gg)}_{rs}\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\dots)|^2 = \\
&\quad = -(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{2}{s_{ir}}\,\frac{1}{z_r}\boldsymbol{T}_i^2\,\sum_j\sum_{l\neq j}\frac{1}{2}\mathcal{S}_{jl}(s)|\mathcal{M}^{(0)}_{m;(j,l)}(p_i,\dots)|^2\,. \qquad (4.46)
\end{aligned}$$

Thus, in the doubly-soft region, eqs. (4.45) and (4.46) differ by subleading terms ($z_i = 1 - z_r = 1 + \mathrm{O}(\lambda)$).

[2]Eq. (4.11) and the azimuth-dependent terms in eq. (4.12) and in the triple-gluon splitting function do not have leading singular terms in the doubly-soft limit, as expected.

In addition to the pairwise overlapping doubly-unresolved regions, there are two regions where three limits overlap. The first one is the overlap of the triply-collinear, the soft-collinear and the doubly-soft emissions. In this region,

$$\begin{aligned}
&\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \frac{2}{s_{ir}}\frac{z_i}{z_r}\boldsymbol{T}_i^2\,\frac{2}{s_{is}}\frac{z_i}{z_s}\boldsymbol{T}_i^2\,|\mathcal{M}_m^{(0)}(p_i,\dots)|^2 \,.
\end{aligned} \tag{4.47}$$

The other region of triple overlap is that of the doubly-collinear, the soft-collinear and the doubly-soft emissions, where

$$\begin{aligned}
&\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \frac{2}{s_{ir}}\frac{z_i}{z_r}\boldsymbol{T}_i^2\,\frac{2}{s_{js}}\frac{z_j}{z_s}\boldsymbol{T}_i^2\,|\mathcal{M}_m^{(0)}(p_i,p_j,\dots)|^2 \,.
\end{aligned} \tag{4.48}$$

Using eqs. (4.31)–(4.48), we can now easily check the following relations,

$$\mathbf{CS}_{ir;s}\Big(\mathbf{C}_{irs} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_s,p_r,\dots)|^2 = 0\,, \tag{4.49}$$

$$\mathbf{CS}_{ir;s}\Big(\mathbf{C}_{ir;js} - \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = 0\,, \tag{4.50}$$

$$\mathbf{CS}_{ir;s}\Big(\mathbf{C}_{irs}\mathbf{S}_{rs} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = 0\,, \tag{4.51}$$

$$\mathbf{CS}_{ir;s}\Big(\mathbf{C}_{ir;js}\mathbf{S}_{rs} - \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = 0\,, \tag{4.52}$$

$$\mathbf{S}_{rs}\Big(\mathbf{C}_{irs} - \mathbf{C}_{irs}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = 0\,, \tag{4.53}$$

$$\mathbf{S}_{rs}\Big(\mathbf{C}_{ir;js} - \mathbf{C}_{ir;js}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = 0\,, \tag{4.54}$$

$$\mathbf{S}_{rs}\Big(\mathbf{CS}_{ir;s} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = 0\,, \tag{4.55}$$

$$\mathbf{S}_{rs}\Big(\mathbf{C}_{irs}\mathbf{CS}_{ir;s} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = 0\,, \tag{4.56}$$

$$\mathbf{S}_{rs}\Big(\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} - \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big)|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = 0\,. \tag{4.57}$$

With eqs. (4.49)–(4.57) at hand it is not difficult to verify that for the subtraction term,

$$\begin{aligned}
\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_r\sum_{s\neq r}\Bigg\{&\sum_{i\neq r,s}\Bigg[\frac{1}{6}\,\mathbf{C}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\,\mathbf{C}_{ir;js} + \\
&+\frac{1}{2}\left(\mathbf{CS}_{ir;s} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s} - \sum_{j\neq i,r,s}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\right)\Bigg] + \\
&+\frac{1}{2}\,\mathbf{S}_{rs} - \sum_{i\neq r,s}\Bigg[\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \mathbf{C}_{irs}\left(\frac{1}{2}\,\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right) + \\
&+\sum_{j\neq i,r,s}\mathbf{C}_{ir;js}\left(\frac{1}{2}\,\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right)\Bigg]\Bigg\}\times \\
&\times|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2\,,
\end{aligned} \tag{4.58}$$

JHEP06(2005)024

the following equations hold,

$$\mathbf{C}_{irs}(1-\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,, \tag{4.59}$$

$$\mathbf{C}_{ir;js}(1-\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,, \tag{4.60}$$

$$\mathbf{CS}_{ir;s}(1-\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,, \tag{4.61}$$

$$\mathbf{S}_{rs}(1-\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,. \tag{4.62}$$

Therefore, eq. (4.58) is free of double and triple subtractions. Instead of eq. (4.58), the subtraction terms that can be derived from antennae factorization, as proposed in refs. [16, 29, 30], can also be used for regularizing the infra-red divergencies in the doubly-unresolved regions. The advantage of antennae subtractions over our proposal is that those avoid the problem of defining the momentum fractions. Note however, that contrary to the original antennae factorization terms of ref. [22], the spin-polarization of the parent parton in the collinear regions is not yet taken into account in refs. [16, 29, 30]. The spin-averaged antennae terms can only be used in a hybrid slicing-subtraction method as applied in ref. [23].

We can simplify eq. (4.58) by removing the terms that contain triple limits. Comparing eqs. (4.43) and (4.48) we see that in the last term in eq. (4.58)

$$\mathbf{C}_{ir;js}\left(\frac{1}{2}\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right)|\mathcal{M}_{m+2}^{(0)}|^2 = -\mathbf{C}_{ir;js}\frac{1}{2}\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}|^2\,. \tag{4.63}$$

Furthermore, comparing eqs. (4.40) and (4.41) to eq. (4.47), and using eq. (4.37), and observing that the soft-collinear limit of eq. (4.41) is zero, we see that

$$\begin{aligned}\mathbf{C}_{irs}&\left(\frac{1}{2}\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right)|\mathcal{M}_{m+2}^{(0)}|^2 = \\ &= \frac{1}{2}\mathbf{C}_{irs}\left[\left(\mathbf{S}_{rs}^{(\mathrm{nab})} - \mathbf{S}_{rs}^{(\mathrm{ab})}\right)\delta_{f_r g}\,\delta_{f_s g} + \mathbf{S}_{rs}\,\delta_{f_r q}\,\delta_{f_s \bar{q}}\right]|\mathcal{M}_{m+2}^{(0)}|^2\,.\end{aligned} \tag{4.64}$$

This equation shows that we can simplify the notation by introducing

$$\mathbf{S}_{rs}^{(\mathrm{A})} \equiv \mathbf{S}_{rs}^{(\mathrm{ab})}\,\delta_{f_r g}\,\delta_{f_s g} \tag{4.65}$$

and

$$\mathbf{S}_{rs}^{(\mathrm{N})} \equiv \mathbf{S}_{rs}^{(\mathrm{nab})}\,\delta_{f_r g}\,\delta_{f_s g} + \mathbf{S}_{rs}\,\delta_{f_r q}\,\delta_{f_s \bar{q}}\,. \tag{4.66}$$

With this notation $\mathbf{S}_{rs} = \mathbf{S}_{rs}^{(\mathrm{A})} + \mathbf{S}_{rs}^{(\mathrm{N})}$.

5. Matching the singly- and doubly-unresolved limits

Eqns. (4.59)–(4.62) ensure that the difference $|\mathcal{M}_{m+2}^{(0)}|^2 - \mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$ does not contain leading singularities (therefore, it is integrable) over those doubly-unresolved parts of the phase space where the factorized matrix elements $|\mathcal{M}_m^{(0)}\rangle$ in the subtraction terms are finite. Nevertheless, the regularized squared matrix element, $|\mathcal{M}_{m+2}^{(0)}|^2 - \mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$, contains leading singularities over the singly-unresolved parts of the phase space. We have a similar problem concerning the subtraction term $\mathbf{A}_1|\mathcal{M}_{m+2}^{(0)}|^2$ that is used for regularizing the

JHEP06(2005)024

singly-unresolved limits, namely, $\mathbf{A}_1|\mathcal{M}_{m+2}^{(0)}|^2$ becomes singular over the doubly-unresolved parts of the phase space. In a general subtraction scheme one wishes to regularize these unwanted singularities, introduced by the subtraction terms, by defining further subtractions that take into account the overlap of the singly- and doubly-unresolved regions. The only consistent way to achieve that is to introduce a third subtraction term $-\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2$ that simultaneously cancels all unwanted singularities in both the singly- and doubly-unresolved parts of the phase space. As a result, the combination

$$(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 \tag{5.1}$$

may serve as a NNLO subtraction term that regularizes the squared matrix element in all relevant unresolved regions of the phase space, i.e. the following equations must hold,

$$\mathbf{C}_{ir}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.2}$$

$$\mathbf{S}_{r}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{S}_{r}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.3}$$

$$\mathbf{C}_{irs}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.4}$$

$$\mathbf{C}_{ir;js}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{C}_{ir;js}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.5}$$

$$\mathbf{CS}_{ir;s}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.6}$$

$$\mathbf{S}_{rs}(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}|^2\,. \tag{5.7}$$

In order to prove eqs. (5.2)–(5.7), we define $\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2$ as $\mathbf{A}_1\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$. Explicitly,

$$\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_t \left[\mathbf{S}_t\mathbf{A}_2 + \sum_{k\neq t}\frac{1}{2}\,\mathbf{C}_{kt}\mathbf{A}_2 - \sum_{k\neq t}\mathbf{C}_{kt}\mathbf{S}_t\mathbf{A}_2\right]|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{5.8}$$

where

$$\begin{aligned}\mathbf{S}_t\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_{r\neq t}\Bigg\{\sum_{i\neq r,t}\Bigg[&\frac{1}{2}\,(\mathbf{S}_t\mathbf{C}_{irt} + \mathbf{S}_t\mathbf{CS}_{ir;t} - \mathbf{S}_t\mathbf{C}_{irt}\mathbf{CS}_{ir;t}) - \\ &- \mathbf{S}_t\mathbf{CS}_{ir;t}\mathbf{S}_{rt} - \mathbf{S}_t\mathbf{C}_{irt}\mathbf{S}_{rt}^{(\mathrm{nab})}\Bigg] + \mathbf{S}_t\mathbf{S}_{rt}\Bigg\}|\mathcal{M}_{m+2}^{(0)}|^2\,,\end{aligned} \tag{5.9}$$

$$\begin{aligned}\mathbf{C}_{kt}\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \Bigg\{\mathbf{C}_{kt}\mathbf{S}_{kt}^{(\mathrm{N})} + \sum_{r\neq k,t}\Bigg[&\sum_{i\neq r,k,t}\left(\frac{1}{2}\,\mathbf{C}_{kt}\mathbf{C}_{ir;kt} - \mathbf{C}_{kt}\mathbf{C}_{ir;kt}\mathbf{CS}_{kt;r}\right) + \\ &+ \mathbf{C}_{kt}\mathbf{C}_{ktr} + \mathbf{C}_{kt}\mathbf{CS}_{kt;r} - \\ &- \mathbf{C}_{kt}\mathbf{C}_{ktr}\mathbf{CS}_{kt;r} - \mathbf{C}_{kt}\mathbf{C}_{rkt}\mathbf{S}_{kt}^{(\mathrm{N})}\Bigg]\Bigg\}|\mathcal{M}_{m+2}^{(0)}|^2\,,\end{aligned} \tag{5.10}$$

and finally

$$\begin{aligned}\mathbf{C}_{kt}\mathbf{S}_t\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \Bigg\{&\mathbf{C}_{kt}\mathbf{S}_t\mathbf{S}_{kt}^{(\mathrm{nab})} + \\ &+ \sum_{r\neq k,t}\Bigg[\sum_{i\neq r,k,t}\left(\frac{1}{2}\,\mathbf{C}_{kt}\mathbf{S}_t\mathbf{CS}_{ir;t} - \mathbf{C}_{kt}\mathbf{S}_t\mathbf{CS}_{ir;t}\mathbf{S}_{rt}\right) + \\ &\quad + \mathbf{C}_{kt}\mathbf{S}_t\mathbf{C}_{krt} + \mathbf{C}_{kt}\mathbf{S}_t\mathbf{S}_{rt} - \mathbf{C}_{kt}\mathbf{S}_t\mathbf{CS}_{kr;t}\mathbf{S}_{rt} - \\ &\quad - \mathbf{C}_{kt}\mathbf{S}_t\mathbf{C}_{rkt}\mathbf{S}_{kt}^{(\mathrm{nab})}\Bigg]\Bigg\}|\mathcal{M}_{m+2}^{(0)}|^2\,.\end{aligned} \tag{5.11}$$

JHEP06(2005)024

Note that the direct calculation of the terms on the right-hand side of eq. (5.8) yields significantly larger expressions than those in eqs. (5.9)–(5.11). In section 7 we shall derive some factorization formulae we used in obtaining eqs. (5.9)–(5.11). In fact, all the terms of eqs. (5.9)–(5.11) are singly-unresolved limits of the factorization formulae for doubly-unresolved emission. Most of them can be computed easily using the explicit expressions of the previous sections. The only non-trivial ones are those that are related to the *iterated singly-unresolved limits*, which we discuss first.

6. Iterated singly-unresolved limits

In this section we derive the factorization formulae valid in the singly-unresolved limits of eqs. (3.10) and (3.13) that we call the iterated singly-unresolved limits. Thus we give the explicit expressions of the various terms in eqs. (3.23) and (3.24). *All the factorization formulae presented in this section are valid for helicity- and colour-correlated squared matrix elements in eqs. (3.10) and (3.13) written in terms of on-shell momenta.*

6.1 Collinear limit of the soft-factorization formula

In the derivation of the collinear limit of the colour-correlated squared matrix element $|\mathcal{M}^{(0)}_{m+1;(j,l)}|^2$ of eq. (3.11), we have to distinguish two cases: (i) the partons of the collinear pair i, r are different from partons j and l; (ii) one parton of the collinear pair is either j or l, the other is different. In principle there is a third case, when j and l become collinear. The corresponding $1/s_{jl}$ pole is suppressed by the s_{jl} factor in the eikonal factor that multiplies $|\mathcal{M}^{(0)}_{m+1;(j,l)}|^2$, therefore, the contribution is subleading. Also, $|\mathcal{M}^{(0)}_{m+1;(j,l)}|^2$ does not have a leading singularity when i and r are both quarks or antiquarks.

In the first case, the colour-charge operators do not influence the computation of the collinear limit, $|\mathcal{M}^{(0)}_{m+1;(j,l)}|^2$ factorizes on the collinear poles as $|\mathcal{M}^{(0)}_{m+1}|^2$ does. In the second case, due to coherent soft-gluon emission from unresolved partons, only the sum of $|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2 + |\mathcal{M}^{(0)}_{m+1;(r,l)}|^2$ (or $|\mathcal{M}^{(0)}_{m+1;(j,i)}|^2 + |\mathcal{M}^{(0)}_{m+1;(j,r)}|^2$) factorizes as

$$\mathcal{S}_{il}(s)|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2 + \mathcal{S}_{rl}(s)|\mathcal{M}^{(0)}_{m+1;(r,l)}|^2 \simeq 8\pi\alpha_s\mu^{2\varepsilon}\,\mathcal{S}_{(ir)l}(s)\,\frac{1}{s_{ir}}\,\langle\mathcal{M}^{(0)}_m|\,\hat{P}^{(0)}_{f_if_r}\,\boldsymbol{T}_{ir}\cdot\boldsymbol{T}_l\,|\mathcal{M}^{(0)}_m\rangle\,, \tag{6.1}$$

where $\boldsymbol{T}_{ir} = \boldsymbol{T}_i + \boldsymbol{T}_r$ and $\mathcal{S}_{(ir)j}(s) = \mathcal{S}_{j(ir)}(s)$ is defined in eq. (4.18). Collecting all the terms, in the collinear limit $i||r$ we obtain

$$\begin{aligned}
&\mathbf{C}_{ir}\mathbf{S}_s|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = -(8\pi\alpha_s\mu^{2\varepsilon})^2\,\times \\
&\qquad \times\frac{1}{s_{ir}}\Bigg\{\sum_{j,l\neq(ir)}\frac{1}{2}\mathcal{S}_{jl}(s)\,\langle\mathcal{M}^{(0)}_m(p_{ir},\ldots)|\,\hat{P}^{(0)}_{f_if_r}(z_i,z_r)\,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\,|\mathcal{M}^{(0)}_m(p_{ir},\ldots)\rangle\,+ \\
&\qquad\qquad + \sum_{l\neq(ir)}\mathcal{S}_{(ir)l}(s)\,\langle\mathcal{M}^{(0)}_m(p_{ir},\ldots)|\,\hat{P}^{(0)}_{f_if_r}(z_i,z_r)\,\boldsymbol{T}_{ir}\cdot\boldsymbol{T}_l\,|\mathcal{M}^{(0)}_m(p_{ir},\ldots)\rangle\Bigg\}\,,
\end{aligned} \tag{6.2}$$

that is,

$$\mathbf{C}_{ir}\mathbf{S}_s|\mathcal{M}^{(0)}_{m+2}|^2 = \mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}|^2\,. \tag{6.3}$$

JHEP06(2005)024

6.2 Soft limit of the soft-factorization formula

We are interested in the limit of the right hand side of eq. (3.11) as a gluon of momentum $p_s = \lambda q_s$ becomes soft ($\lambda \to 0$ at fixed q_s). The soft limit of the colour-correlated squared matrix elements, $|\mathcal{M}^{(0)}_{m+1;(i,k)}|^2$, can be obtained in a similar way as that of the squared matrix element. We have to distinguish two cases: (i) s is different from i and k, (ii) either i or k gets soft.

Let us start with case (i), when we can immediately use the well known factorization of the matrix element $|\mathcal{M}^{(0)}_{m+1}(p_s,\ldots)\rangle$,

$$\langle\nu|\mathcal{M}^{(0)}_{m+1}(p_s,\ldots)\rangle \simeq g_{\mathrm{s}}\mu^{\varepsilon}\boldsymbol{J}^{\nu}(p_s)|\mathcal{M}^{(0)}_{m}(\ldots)\rangle\,, \tag{6.4}$$

where we neglected all contributions that are less singular than $1/\lambda$. The factor $\boldsymbol{J}^{\nu}(p_s)$ is the eikonal current for the emission of the soft gluon s,

$$\boldsymbol{J}^{\nu}(p_s) = \sum_j \boldsymbol{T}_j \frac{p_j^{\nu}}{p_j\cdot p_s}\,. \tag{6.5}$$

We substitute eq. (6.4) into $|\mathcal{M}^{(0)}_{m+1;(i,k)}(p_s,\ldots)|^2$ and obtain

$$\mathbf{S}_s|\mathcal{M}^{(0)}_{m+1;(i,k)}(p_s,\ldots)|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_{j,l}\frac{1}{2}\mathcal{S}_{jl}(s)\langle\mathcal{M}^{(0)}_{m}(\ldots)|T^a_jT^b_iT^b_kT^a_l|\mathcal{M}^{(0)}_{m}(\ldots)\rangle\,. \tag{6.6}$$

We can use the colour algebra to derive the identity

$$\begin{aligned}2T^a_jT^b_iT^b_kT^a_l &= \{\boldsymbol{T}_j\cdot\boldsymbol{T}_l,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\} - \\ &- \mathrm{i}f_{abc}(T^a_jT^c_iT^b_k\delta_{li} + T^a_jT^b_iT^c_k\delta_{lk} + T^a_lT^c_iT^b_k\delta_{jk} + T^a_lT^b_iT^c_k\delta_{ji}) + \\ &+ C_{\mathrm{A}}\boldsymbol{T}_i\boldsymbol{T}_k(\delta_{li}\delta_{jk} - \delta_{lk}\delta_{jk} - \delta_{li}\delta_{ji} + \delta_{lk}\delta_{ji})\,,\end{aligned} \tag{6.7}$$

which we substitute into eq. (6.6). Performing the summation over j and l, we find that the terms proportional to f_{abc} in eq. (6.7) are symmetric in b and c, while f_{abc} is antisymmetric, therefore, the sum of these terms is zero. The remaining terms yield

$$\begin{aligned}&\mathbf{S}_s|\mathcal{M}^{(0)}_{m+1;(i,k)}(p_s,\ldots)|^2 = \\ &\quad = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{4}\left[\sum_{j,l}\mathcal{S}_{jl}(s)|\mathcal{M}^{(0)}_{m;(i,k)(j,l)}(\ldots)|^2 + 2C_{\mathrm{A}}\,\mathcal{S}_{ik}(s)|\mathcal{M}^{(0)}_{m;(i,k)}(\ldots)|^2\right].\end{aligned} \tag{6.8}$$

In the second case, when for instance gluon i gets soft, we first use colour conservation, $\boldsymbol{T}_s = -\sum_{l\neq s}\boldsymbol{T}_l$ to write

$$|\mathcal{M}^{(0)}_{m+1;(s,k)}(p_s,\ldots)|^2 = -\sum_{l\neq s}|\mathcal{M}^{(0)}_{m+1;(l,k)}(p_s,\ldots)|^2\,. \tag{6.9}$$

We obtain the factorization formulae for the $|\mathcal{M}^{(0)}_{m+1;(l,k)}|^2$ colour-correlated matrix elements the same way as we did in case (i), and substitute those into eq. (6.9),

$$\begin{aligned}&\mathbf{S}_s|\mathcal{M}^{(0)}_{m+1;(s,k)}(p_s,\ldots)|^2 = \\ &\quad = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{4}\sum_{l}\left[\sum_{j,j'}\mathcal{S}_{jj'}(s)|\mathcal{M}^{(0)}_{m;(l,k)(j,j')}(\ldots)|^2 + 2C_{\mathrm{A}}\,\mathcal{S}_{lk}(s)|\mathcal{M}^{(0)}_{m;(l,k)}(\ldots)|^2\right].\end{aligned} \tag{6.10}$$

JHEP06(2005)024

In the first term we can perform the summation over l using colour conservation, and find that the sum is zero. Therefore,

$$\mathbf{S}_s|\mathcal{M}^{(0)}_{m+1;(s,k)}(p_s,\ldots)|^2 = 8\pi\alpha_{\rm s}\mu^{2\varepsilon}\frac{1}{2}C_{\rm A}\sum_l \mathcal{S}_{lk}(s)\left|\mathcal{M}^{(0)}_{m;(l,k)}(\ldots)\right|^2 . \tag{6.11}$$

We check explicitly the validity of eqs. (6.8) and (6.11) in appendix A using the known colour-correlated squared matrix elements for the $e^+e^- \to q\bar{q} + ng$ ($n = 1$, 2) processes.

We can now easily compute the soft-factorization formula for the right-hand side of eq. (3.11). We decompose it as

$$\sum_{i,k}\frac{1}{2}\mathcal{S}_{ik}(r)\left|\mathcal{M}^{(0)}_{m+1;(i,k)}\right|^2 = \sum_{i,k\neq s}\frac{1}{2}\mathcal{S}_{ik}(r)\left|\mathcal{M}^{(0)}_{m+1;(i,k)}\right|^2 + \sum_k \mathcal{S}_{sk}(r)\left|\mathcal{M}^{(0)}_{m+1;(s,k)}\right|^2 . \tag{6.12}$$

In the soft limit all the eikonal factors are finite, so we can simply substitute eqs. (6.8) and (6.11), and obtain

$$\begin{aligned}\mathbf{S}_s\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}(p_r,p_s,\ldots)|^2 &=\\ &= (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{4}\Bigg[\sum_{i,k,j,l}\frac{1}{2}\mathcal{S}_{ik}(r)\mathcal{S}_{jl}(s)|\mathcal{M}^{(0)}_{m;(i,k)(j,l)}(\ldots)|^2 - \\ &\qquad - C_{\rm A}\sum_{i,k=1}^{m}\mathcal{S}^{(\rm s.o.)}_{ik}(r,s)|\mathcal{M}^{(0)}_{m;(i,k)}(\ldots)|^2\Bigg],\end{aligned} \tag{6.13}$$

where

$$\mathcal{S}^{(\rm s.o.)}_{ik}(r,s) = \mathcal{S}_{ik}(s)\left(\mathcal{S}_{is}(r) + \mathcal{S}_{ks}(r) - \mathcal{S}_{ik}(r)\right) \tag{6.14}$$

is the two-gluon soft function in the strongly-ordered approximation. If either r or s is a quark, then the result is zero.

Finally, we consider the region where the soft momentum p_s becomes also collinear to the momentum of another parton j. In this region, we have

$$\frac{1}{4}\mathcal{S}_{jk}(s)\left(\mathcal{S}_{js}(r) + \mathcal{S}_{ks}(r) - \mathcal{S}_{jk}(r)\right) \simeq \frac{s_{jk}}{s_{jr}s_{rs}s_{sk}} \simeq {\rm O}\left(\frac{1}{\lambda^2}\right), \tag{6.15}$$

thus the term proportional to $C_{\rm A}$ is subleading. The eikonal factor $\mathcal{S}_{jl}(s)$ becomes independent of l,

$$\mathcal{S}_{jl}(s) \simeq \frac{2}{s_{js}}\frac{z_j}{z_s}, \tag{6.16}$$

therefore, using colour conservation, we can perform the summation over l in the first sum on the right-hand side of eq. (6.13), and obtain

$$\mathbf{C}_{js}\mathbf{S}_s\mathbf{S}_r\left|\mathcal{M}^{(0)}_{m+2}(p_r,p_s,\ldots)\right|^2 = -(8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{2}{s_{js}}\frac{z_j}{z_s}\boldsymbol{T}_j^2\sum_{i,k}\frac{1}{2}\mathcal{S}_{ik}(r)\left|\mathcal{M}^{(0)}_{m;(i,k)}(\ldots)\right|^2 . \tag{6.17}$$

Comparing this result to eq. (6.2), using eq. (6.3) and

$$\mathbf{S}_s\mathbf{C}\mathbf{S}_{js;r}|\mathcal{M}^{(0)}_{m+2}(p_j,p_r,p_s,\ldots)|^2 = 8\pi\alpha_{\rm s}\mu^{2\varepsilon}\,\frac{2}{s_{js}}\,\frac{1}{z_s}\,\boldsymbol{T}_j^2\,\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_i,p_r,\ldots)|^2 , \tag{6.18}$$

we see that

$$\mathbf{S}_s\left(\mathbf{C}_{js} - \mathbf{C}_{js}\mathbf{S}_s\right)\mathbf{S}_r\left|\mathcal{M}^{(0)}_{m+2}(p_r,p_s,\ldots)\right|^2 = 0 . \tag{6.19}$$

JHEP06(2005)024

6.3 Collinear limit of the collinear-factorization formula

Next, we consider the limit of eq. (3.10) when parton s becomes collinear with another parton j. The collinear-factorization formula involves the helicity-dependent tensors,

$$\mathcal{T}^{hh'}_{m+1} = \langle \mathcal{M}^{(0)}_{m+1}(p_{ir}, p_j, p_s, \ldots)|h\rangle\langle h'|\mathcal{M}^{(0)}_{m+1}(p_{ir}, p_j, p_s, \ldots)\rangle\,, \tag{6.20}$$

which are the leading-order $(m+1)$-parton squared matrix elements not summed over the spin polarizations of parton (ir), i.e., if parton (ir) is a gluon, then $-g_{\mu\nu}\mathcal{T}^{\mu\nu}_{m+1} = |\mathcal{M}^{(0)}_{m+1}|^2$, and $\delta_{ss'}\mathcal{T}^{ss'}_{m+1} = |\mathcal{M}^{(0)}_{m+1}|^2$ if (ir) is a quark.

The computation of the collinear limit of the $\mathcal{T}^{hh'}_{m+1}$ tensors follows that for the collinear factorization of the squared matrix element. When $j \neq (ir)$, we obtain

$$\begin{aligned} &\mathbf{C}_{js}\mathcal{T}^{hh'}_{m+1}(p_j, p_s, p_{ir}, \ldots) = \\ &\qquad = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{js}}\,\langle \mathcal{M}^{(0)}_{m}(p_{js}, p_{ir}, \ldots)|h\rangle\,\hat{P}^{(0)}_{f_j f_s}(z_j, z_s, k_{s\perp}; \varepsilon)\,\langle h'|\mathcal{M}^{(0)}_{m}(p_{js}, p_{ir}, \ldots)\rangle\,. \end{aligned} \tag{6.21}$$

If $j = (ir)$, then we find

$$\begin{aligned} &\mathbf{C}_{(ir)s}\mathcal{T}^{hh'}_{m+1}(p_{ir}, p_s, \ldots) = \\ &\qquad = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{(ir)s}}\langle \mathcal{M}^{(0)}_{m}(p_{(ir)s}, \ldots)|\,\hat{P}^{hh'}_{f_{ir} f_s}(z_s, k_{s\perp}; p_{(ir)s}, n_{(ir)s}; \varepsilon)\,|\mathcal{M}^{(0)}_{m}(p_{(ir)s}, \ldots)\rangle\,. \end{aligned} \tag{6.22}$$

The $\hat{P}^{hh'}_{f_i f_j}(z, k_\perp; p, n; \varepsilon)$ kernels (splitting tensors) are related to the d-dimensional Altarelli-Parisi splitting functions by summation over the spin indices h and h',

$$\delta_{hh'}\langle s|\hat{P}^{hh'}_{qg}(z, k_\perp; p, n; \varepsilon)|s'\rangle \;=\; \langle s|\hat{P}^{(0)}_{qg}(1-z, z; \varepsilon)|s'\rangle\,, \tag{6.23}$$

$$\delta_{hh'}\langle \mu|\hat{P}^{hh'}_{q\bar{q}}(z, k_\perp; p, n; \varepsilon)|\nu\rangle \;=\; \langle \mu|\hat{P}^{(0)}_{q\bar{q}}(1-z, z, k_\perp; \varepsilon)|\nu\rangle\,, \tag{6.24}$$

$$d_{\alpha\beta}(p,n)\langle s|\hat{P}^{\alpha\beta}_{gq}(z, k_\perp; p, n; \varepsilon)|s'\rangle \;=\; \langle s|\hat{P}^{(0)}_{qg}(z, 1-z; \varepsilon)|s'\rangle\,, \tag{6.25}$$

$$d_{\alpha\beta}(p,n)\langle \mu|\hat{P}^{\alpha\beta}_{gg}(z, 1-z, k_\perp; p, n; \varepsilon)|\nu\rangle \;=\; \langle \mu|\hat{P}^{(0)}_{gg}(z, 1-z, k_\perp; \varepsilon)|\nu\rangle\,. \tag{6.26}$$

Here $d_{\alpha\beta}(p,n)$ is the gluon polarization tensor for physical polarizations,

$$d_{\alpha\beta}(p,n) = -g_{\alpha\beta} + \frac{p_\alpha n_\beta + p_\beta n_\alpha}{p\cdot n}\,. \tag{6.27}$$

In order to obtain the collinear limit of the collinear-factorization formula, we insert eqs. (6.21) and (6.22) into eq. (3.4). For $j \neq (ir)$ we obtain

$$\begin{aligned} &\mathbf{C}_{js}\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_j, p_s, \ldots)|^2 = \\ &\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\,\frac{1}{s_{ir}s_{js}}\times \\ &\qquad \times \langle \mathcal{M}^{(0)}_{m}(p_{ir}, p_{js}, \ldots)|\,\hat{P}^{(0)}_{f_i f_r}(z_i, z_r, k_{r\perp}; \varepsilon)\,\hat{P}^{(0)}_{f_j f_s}(z_j, z_s, k_{j\perp}; \varepsilon)\,|\mathcal{M}^{(0)}_{m}(p_{ir}, p_{js}, \ldots)\rangle\,. \end{aligned} \tag{6.28}$$

The quark splitting function, $\langle h|\hat{P}^{(0)}_{qg}(z_q, z_g, k_\perp; \varepsilon)|h'\rangle$, is diagonal in the spin indices (see eq. (3.7)), therefore,

$$\langle h|\hat{P}^{(0)}_{qg}(z_q, z_g, k_{r\perp}; \varepsilon)|h'\rangle\,\hat{P}^{hh'}_{qf_s}(z_s, k_{s\perp}; p_{(ir)s}, n_{(ir)s}; \varepsilon) = P^{(0)}_{qg}(z_q, z_g; \varepsilon)\hat{P}^{(0)}_{qf_s}(1-z_s, z_s; \varepsilon)\,. \tag{6.29}$$

JHEP06(2005)024

As a result, the factorization formula in the case of parton $j = (ir)$ being a quark is

$$
\begin{aligned}
&\mathbf{C}_{(ir)s}\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\,\frac{1}{s_{ir}s_{(ir)s}}\,P_{qg}^{(0)}(z_q,z_g,k_{r\perp};\varepsilon)\times \\
&\qquad \times \langle\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)|\,\hat{P}_{qf_s}^{(0)}(1-z_s,z_s,k_{ir\perp};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)\rangle.
\end{aligned}
\tag{6.30}
$$

Thus we need explicit expressions only for the $\hat{P}_{gf_s}^{\alpha\beta}$ splitting tensors,

$$
\begin{aligned}
&\langle s|\hat{P}_{gq}^{\alpha\beta}(z,k_\perp;p,n;\varepsilon)|s'\rangle = \\
&\quad = C_{\rm F}\,\delta_{ss'}\left[-\frac{1-z}{2}g^{\alpha\beta}-2\frac{z}{1-z}\frac{k_\perp^\alpha k_\perp^\beta}{k_\perp^2}+(1-z)\frac{p^\alpha n^\beta+p^\beta n^\alpha}{2p\cdot n}\right]+\cdots\,, \qquad (6.31)\\
&\langle\mu|\hat{P}_{gg}^{\alpha\beta}(z,k_\perp;p,n;\varepsilon)|\nu\rangle = \\
&\quad = 2C_{\rm A}\left[\frac{1-z}{z}g^{\alpha\mu}g^{\beta\nu}+\frac{z}{1-z}g^{\mu\nu}\frac{k_\perp^\alpha k_\perp^\beta}{k_\perp^2}-z(1-z)d^{\alpha\beta}(p,n)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]+\cdots\,. \qquad (6.32)
\end{aligned}
$$

In eqs. (6.31) and (6.32) the ellipses mean neglected terms that do not contribute to the contraction of the splitting functions and splitting tensors

$$
\langle\alpha|\hat{P}_{gg}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)|\beta\rangle\,\hat{P}_{gf_s}^{\alpha\beta}(z_s,k_{s\perp};p_{(ir)s},n_{(ir)s};\varepsilon)
\tag{6.33}
$$

(summation over the gluon polarization indices α and β is understood), which appear in the factorization formula for parton $j = (ir)$ being a gluon,

$$
\begin{aligned}
&\mathbf{C}_{(ir)s}\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{s_{ir}}\frac{1}{s_{(ir)s}}\langle\alpha|\hat{P}_{gg}^{(0)}(z_i,z_r,k_{r\perp};\varepsilon)|\beta\rangle\times \\
&\qquad \times\langle\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)|\,\hat{P}_{gf_s}^{\alpha\beta}(z_s,k_{s\perp};p_{(ir)s},n_{(ir)s};\varepsilon)\,|\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)\rangle\,.
\end{aligned}
\tag{6.34}
$$

In appendix B, we use the known $\mathcal{T}^{\mu\nu}$ tensor for the process $e^+e^-\to q\bar{q}g$ to check the validity of eqs. (6.31) and (6.34) explicitly. In appendix C, we translate eq. (6.34) to a helicity basis and compare the obtained expression to the corresponding one derived using the known collinear factorization properties of helicity amplitudes.

We note that the collinear-factorization formula for the $\mathcal{T}^{\mu\nu}$ tensor in eq. (6.22) fulfills gauge invariance (only) in the collinear limit,

$$
p_{ir\,\alpha}\langle\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)|\hat{P}_{gf_s}^{\alpha\beta}(z,k_\perp;p,n;\varepsilon)|\mathcal{M}_m^{(0)}(p_{(ir)s},\ldots)\rangle = {\rm O}(k_\perp)\,,
\tag{6.35}
$$

where $p_{ir}^\alpha = (1-z)p_{(ir)s}^\alpha - k_\perp^\alpha + {\rm O}(k_\perp^2)$, and in obtaining eq. (6.35) for $f_s = g$, we used $p_{(ir)s\,\mu}\mathcal{T}_m^{\mu\nu} = 0$. We kept the gauge terms in eqs. (6.31) and (6.32) only to maintain this gauge invariance; those terms do not contribute to either eqs. (6.25) and (6.26) or eq. (6.33).

JHEP06(2005)024

6.4 Soft limit of the collinear-factorization formula

As in the case of the soft limit of the soft-factorization formula in section 6.2, in order to obtain the soft limit of the collinear-factorization formula we use the factorization of the matrix element, eq. (6.4), which we substitute into eq. (3.4). If the parent parton of the collinear splitting is a gluon, the factorization formula contains non-trivial spin correlations (see eqs. (3.5) and (3.6)), therefore, we have to distinguish two cases: (i) (ir) is a hard parton; (ii) (ir) is the soft gluon. If the soft gluon is labelled with s, we find that

- if (ir) is a hard parton, then

$$\begin{aligned}
&\mathbf{S}_s\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\dots)|^2 = \\
&\quad = -(8\pi\alpha_s\mu^{2\varepsilon})^2 \times \\
&\qquad \times \frac{1}{s_{ir}}\Bigg\{\sum_{j,l\neq(ir)}\frac{1}{2}\mathcal{S}_{jl}(s)\,\langle\mathcal{M}^{(0)}_m(\dots)|\,\hat{P}^{(0)}_{f_if_r}\,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\,|\mathcal{M}^{(0)}_m(\dots)\rangle + \\
&\qquad\qquad + \sum_{l\neq(ir)}\mathcal{S}_{(ir)l}(s)\,\langle\mathcal{M}^{(0)}_m(\dots)|\,\hat{P}^{(0)}_{f_if_r}\,\boldsymbol{T}_{ir}\cdot\boldsymbol{T}_l\,|\mathcal{M}^{(0)}_m(\dots)\rangle\Bigg\}; \qquad (6.36)
\end{aligned}$$

- if $(ir)=s$ is the soft gluon, then

$$\begin{aligned}
\mathbf{S}_{(ir)}\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\dots)|^2 &= (8\pi\alpha_s\mu^{2\varepsilon})^2 \times \qquad (6.37)\\
&\quad \times\frac{1}{s_{ir}}\sum_{j,l\neq s}\frac{1}{2}\mathcal{S}^{\mu\nu}_{jl}(s)\,\langle\mu|\hat{P}^{(0)}_{f_if_r}|\nu\rangle|\mathcal{M}^{(0)}_{m;(j,l)}(\dots)|^2\,,
\end{aligned}$$

where

$$\mathcal{S}^{\mu\nu}_{jl}(s) = 4\frac{p^\mu_j p^\nu_l}{s_{js}s_{sl}}\,, \qquad (6.38)$$

so $g_{\mu\nu}\mathcal{S}^{\mu\nu}_{jl}(s)=\mathcal{S}_{jl}(s)$. Notice that the right-hand side of eq. (6.36) coincides with that of eq. (6.2),

$$\mathbf{S}_s\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}|^2 = \mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}|^2\,. \qquad (6.39)$$

If the soft momentum p_s becomes collinear to the momentum of another parton j, then eq. (6.36) becomes

$$\begin{aligned}
&\mathbf{C}_{js}\mathbf{S}_s\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,p_j,\dots)|^2 = \\
&\quad = (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{1}{s_{ir}}\,\frac{2}{s_{js}}\,\boldsymbol{T}^2_j\,\frac{z_j}{z_s}\langle\mathcal{M}^{(0)}_m(\dots)|\,\hat{P}^{(0)}_{f_if_r}\,|\mathcal{M}^{(0)}_m(\dots)\rangle\,, \qquad (6.40)
\end{aligned}$$

where j can also be (ir).

If $(ir)=s$ is the soft gluon, then in the $j||s$ limit eq. (6.37) becomes

$$\begin{aligned}
&\mathbf{C}_{j(ir)}\mathbf{S}_{(ir)}\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,\dots)|^2 = \\
&\quad = (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{1}{s_{ir}}\,\frac{2}{s_{j(ir)}}\,\boldsymbol{T}^2_j\,\frac{z_j}{z_{ir}}\,2C_A\left(\frac{z_i}{z_r}+\frac{z_r}{z_i}\right)|\mathcal{M}^{(0)}_m(\dots)|^2 \qquad (6.41)
\end{aligned}$$

JHEP06(2005)024

if i and r are both gluons, and

$$
\begin{aligned}
&\mathbf{C}_{j(ir)}\mathbf{S}_{(ir)_g}\mathbf{C}_{i_q r_{\bar{q}}}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,\dots)|^2 = \\
&\qquad = (8\pi\alpha_s\mu^{2\varepsilon})^2 \frac{1}{s_{ir}}\frac{2}{s_{j(ir)}}\boldsymbol{T}_j^2\frac{z_j}{z_{ir}}T_{\mathrm{R}}|\mathcal{M}_m^{(0)}(\dots)|^2
\end{aligned}
\tag{6.42}
$$

if i and r are quark and antiquark, where we used $k_\perp \cdot p_{(ir)} = 0$. Comparing eq. (6.28) to eq. (6.40) and eq. (6.34) to eqs. (6.41) and (6.42), we find that

$$
\mathbf{S}_s\left(\mathbf{C}_{js} - \mathbf{C}_{js}\mathbf{S}_s\right)\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,p_j,\dots)|^2 = 0\,, \tag{6.43}
$$

no matter whether or not the soft parton s is (ir), or whether the soft parton becomes collinear with the hard parton (ir).

We remark that the treatment of colour in the derivations of these iterated singly-unresolved limits was different for the spin-correlated matrix elements in eq. (3.4) from that for the colour-correlated matrix elements in eq. (3.11). This means that soft factorization formulae do not exist for the simultaneously spin- and colour-correlated squared matrix elements.

6.5 Doubly-unresolved limits in the strongly-ordered approximation

The factorization formulae of the iterated limits coincide with those valid in the strongly-ordered doubly-unresolved parts of the phase space. This statement can be used in proving eqs. (5.2)–(5.7), therefore, we discuss the strongly-ordered limits briefly.

First we consider the strongly-ordered doubly-collinear limit, which can occur in two ways. One is defined by the Sudakov parametrization of the momenta of two separate splitting pairs $ir \to i + r$ and $js \to j + s$ as in section 4.2, but with the strongly-ordered rescaling,

$$
k_{r\perp} \to \lambda_r k_{r\perp}\,, \qquad k_{s\perp} \to \lambda_s k_{s\perp}\,, \tag{6.44}
$$

and studying the limit λ_r, $\lambda_s \to 0$, with $\lambda_s \gg \lambda_r$. Keeping terms that are of $\mathrm{O}[1/(\lambda_r^2\lambda_s^2)]$, the squared matrix element $|\mathcal{M}_{m+2}|^2$ behaves precisely as described by eq. (4.15).

The other strongly-ordered doubly-collinear limit occurs when there are two subsequent collinear splittings from the same hard parton, $irs \to ir+s \to i+r+s$, which is the strongly-ordered limit of the triply-collinear emission discussed in section 4.1. In order to describe the $ir \to i + r$ collinear splitting, we introduce the following Sudakov parametrization of the parton momenta,

$$
\begin{aligned}
p_i^\mu &= \zeta_i p_{ir}^\mu - \kappa^\mu - \frac{\kappa^2}{\zeta_i}\frac{n_{ir}^\mu}{2p_{ir}n_{ir}}\,, \qquad p_r^\mu = \zeta_r p_{ir}^\mu + \kappa^\mu - \frac{\kappa^2}{\zeta_r}\frac{n_{ir}^\mu}{2p_{ir}n_{ir}}\,, \\
p_{ir}^\mu &= (1-z_s)p_{(ir)s}^\mu - k_{s\perp}^\mu - \frac{k_{s\perp}^2}{1-z_s}\frac{n_{(ir)s}^\mu}{2p_{(ir)s}n_{(ir)s}}\,,
\end{aligned}
\tag{6.45}
$$

where p_{ir}^μ is a light-like momentum that points towards the collinear direction of partons i and r, n_{ir}^μ is an auxiliary light-like vector that is orthogonal to the transverse component of the third parton ($n_{ir}k_{s\perp} = 0$) and κ are the momentum components that are orthogonal

JHEP06(2005)024

to both p_{ir} and n_{ir}. Without loss of generality, we may choose $n_{ir} = n_{(ir)s}$. The other vectors are defined in the same way as after eq. (4.1). Comparing the two parametrizations of momenta p_i and p_r in eqs. (4.1) and (6.45), we deduce that

$$\zeta_j = \frac{z_j}{1-z_s}\,, \qquad j = i, r \tag{6.46}$$

and

$$k_{i\perp} = -\zeta_i k_{s\perp} - \kappa\,, \qquad k_{r\perp} = -\zeta_r k_{s\perp} + \kappa\,. \tag{6.47}$$

The strongly-ordered collinear region is identified by performing the rescaling

$$\kappa \to \lambda\kappa\,, \qquad k_{s\perp} \to \lambda_s k_{s\perp} \tag{6.48}$$

and studying the limit λ, $\lambda_s \to 0$, with $\lambda_s \gg \lambda$. In this limit eqs. (4.4)–(4.12), as well as the triple-gluon splitting kernel take much simpler forms. In fact, when neglecting terms that are less singular than $1/(\lambda^2\lambda_s^2)$ the squared matrix element $|\mathcal{M}_{m+2}|^2$ behaves as

$$\begin{aligned}
&\mathbf{C}_{(ir)s}\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{s_{ir}}\frac{1}{s_{(ir)s}}\times \\
&\qquad \times \langle\mathcal{M}^{(0)}_m(p_{irs},\ldots)|\,\hat{P}^{\mathrm{s.o.}}_{f_if_rf_s}(\zeta_i,\zeta_r,\kappa,z_s,k_{s\perp},s_{ir},s_{is},s_{rs};\varepsilon)\,|\mathcal{M}^{(0)}_m(p_{irs},\ldots)\rangle\,,
\end{aligned} \tag{6.49}$$

where the kernels $\hat{P}^{\mathrm{s.o.}}_{f_if_rf_s}$ are those in section 4.1 in the strongly-ordered approximation. Explicitly,

$$\begin{aligned}
&P^{\mathrm{s.o.}}_{q_s\bar{q}'_iq'_r}(\zeta_i,\zeta_r,\kappa,z_s,s_{ir},s_{is},s_{rs};\varepsilon) = \\
&\quad = T_{\mathrm{R}}\left[P^{(0)}_{q_sg}(z_s,1-z_s;\varepsilon) - 2C_{\mathrm{F}}\left(\zeta_i\,\zeta_r\,(1-z_s) + \frac{s^2_{\kappa s}}{s_{ir}s_{(ir)s}}\right)\right],
\end{aligned} \tag{6.50}$$

$$P^{(\mathrm{ab})\,\mathrm{s.o.}}_{q_ig_rg_s}(\zeta_i,\zeta_r,z_s;\varepsilon) = P^{(0)}_{qg_s}(1-z_s,z_s;\varepsilon)P^{(0)}_{q_ig_r}(\zeta_i,\zeta_r;\varepsilon)\,, \tag{6.51}$$

$$\begin{aligned}
&P^{(\mathrm{nab})\,\mathrm{s.o.}}_{q_sg_ig_r}(\zeta_i,\zeta_r,\kappa,z_s,s_{ir},s_{is},s_{rs};\varepsilon) = \\
&\quad = 2C_{\mathrm{A}}\left[P^{(0)}_{q_sg}(z_s,1-z_s;\varepsilon)\left(\frac{\zeta_i}{\zeta_r}+\frac{\zeta_r}{\zeta_i}\right) + C_{\mathrm{F}}\,(1-\varepsilon)\left(\zeta_i\,\zeta_r\,(1-z_s)+\frac{s^2_{\kappa s}}{s_{ir}s_{(ir)s}}\right)\right],
\end{aligned} \tag{6.52}$$

where $s_{\kappa s} = 2\kappa\cdot p_s$. Furthermore,

$$\langle\mu|\hat{P}^{(\mathrm{ab})\,\mathrm{s.o.}}_{g_iq_r\bar{q}_s}(\zeta_i,\zeta_r,z_s,k_{s\perp};\varepsilon)|\nu\rangle = P^{(0)}_{q_rg_i}(\zeta_r,\zeta_i;\varepsilon)\langle\mu|\hat{P}^{(0)}_{\bar{q}q}(z_s,1-z_s,k_{s\perp};\varepsilon)|\nu\rangle\,, \tag{6.53}$$

$$\begin{aligned}
&\langle\mu|\hat{P}^{(\mathrm{nab})\,\mathrm{s.o.}}_{g_sq_i\bar{q}_r}(\zeta_i,\zeta_r,\kappa,z_s,k_{s\perp},s_{ir},s_{is},s_{rs};\varepsilon)|\nu\rangle = \\
&\quad = 2C_{\mathrm{A}}T_{\mathrm{R}}\left[-g^{\mu\nu}\left(\frac{z_s}{1-z_s}+\frac{1-z_s}{z_s}\right)+g^{\mu\nu}\frac{s^2_{\kappa s}}{s_{ir}s_{(ir)s}}+4\zeta_i\,\zeta_r\frac{1-z_s}{z_s}\frac{\kappa^\mu\kappa^\nu}{\kappa^2}\right] - \\
&\qquad - 4C_{\mathrm{A}}(1-\varepsilon)\,z_s\,(1-z_s)P^{(0)}_{q_i\bar{q}_r}(\zeta_i,\zeta_r,\kappa;\varepsilon)\frac{k^\mu_{s\perp}k^\nu_{s\perp}}{k^2_{s\perp}}\,,
\end{aligned} \tag{6.54}$$

$$\begin{aligned}
&\langle\mu|\hat{P}^{\mathrm{s.o.}}_{g_ig_rg_s}(\zeta_i,\zeta_r,\kappa,z_s,k_{s\perp},s_{ir},s_{is},s_{rs};\varepsilon)|\nu\rangle = \\
&\quad = 4C^2_{\mathrm{A}}\left[-g^{\mu\nu}\left(\frac{z_s}{1-z_s}+\frac{1-z_s}{z_s}\right)\left(\frac{\zeta_i}{\zeta_r}+\frac{\zeta_r}{\zeta_i}\right) - g^{\mu\nu}\frac{1-\varepsilon}{2}\frac{s^2_{\kappa s}}{s_{ir}s_{(ir)s}} - \right.
\end{aligned}$$

JHEP06(2005)024

$$
\left. - 2(1-\varepsilon)\, \zeta_i\, \zeta_r\, \frac{1-z_s}{z_s} \frac{\kappa^\mu \kappa^\nu}{\kappa^2} \right] -
$$

$$
- 4C_{\mathrm{A}}\, (1-\varepsilon)\, z_s\, (1-z_s)\, P_{gg}^{(0)}\, (\zeta_i, \zeta_r)\, \frac{k_{s\perp}^\mu k_{s\perp}^\nu}{k_{s\perp}^2} \,. \tag{6.55}
$$

In actual computations we can use

$$
\kappa^\mu = \zeta_r p_i^\mu - \zeta_i p_r^\mu + \mathrm{O}(\kappa^2)\,, \qquad k_{s\perp}^\mu = z_s p_{ir}^\mu - z_{ir} p_s^\mu + \mathrm{O}(k_{s\perp}^2)\,. \tag{6.56}
$$

The $\hat{P}^{\mathrm{s.o.}}_{f_i f_r f_s}$ kernels can also be written in a unified form,

$$
\begin{aligned}
&\langle \chi | \hat{P}^{(0)\,\mathrm{s.o.}}_{f_i f_r f_s}(\zeta_i, \zeta_r, \kappa, z_s, k_{s\perp}, s_{ir}, s_{is}, s_{rs}; \varepsilon) | \chi' \rangle = \\
&\quad = \sum_{h,h'} \langle h | \hat{P}^{(0)}_{f_i f_r}(\zeta_i, \zeta_r, \kappa; \varepsilon) | h' \rangle\, \langle \chi | \hat{P}^{hh'}_{f_{ir} f_s}(z_s, k_{s\perp}; p_{irs}, n_{irs}; \varepsilon) | \chi' \rangle\,,
\end{aligned} \tag{6.57}
$$

where we used (see appendix B)

$$
4\zeta_i\, \zeta_r\, \frac{z_s}{1-z_s} \frac{(\kappa \cdot k_{s\perp})^2}{\kappa^2 k_{s\perp}^2} = \frac{s_{\kappa s}^2}{s_{ir} s_{(ir)s}} + \mathrm{O}(k_{s\perp})\,. \tag{6.58}
$$

Therefore, eq. (6.49) coincides with eqs. (6.28), (6.30) and (6.34).

The strongly-ordered doubly-soft emission is defined by parametrizing the momenta p_r and p_s as $p_r = \lambda_r q_r$ and $p_s = \lambda_s q_s$, and taking the limit λ_r, $\lambda_s \to 0$, with $\lambda_s \gg \lambda_r$. In this limit the squared matrix element $|\mathcal{M}_{m+2}|^2$ does not have a leading $\mathrm{O}[1/(\lambda_r^2 \lambda_s^2)]$ singularity when a soft $q\bar{q}$-pair is emitted. In the case of the emission of two soft gluons, the singular behaviour of $|\mathcal{M}_{m+2}|^2$ can be written as in eq. (4.21), the sum of an abelian and a non-abelian contribution,

$$
\begin{aligned}
\mathbf{S}_s \mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, p_s, \ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \frac{1}{4} \Bigg(&\frac{1}{2} \sum_{i,j,k,l=1}^{m} \mathcal{S}_{ik}(p_r) \mathcal{S}_{jl}(p_s) |\mathcal{M}^{(0)}_{m,(i,k)(j,l)}|^2 - \\
&- C_{\mathrm{A}} \sum_{i,k=1}^{m} \mathcal{S}^{(\mathrm{s.o.})}_{ik}(p_r, p_s) |\mathcal{M}^{(0)}_{m,(i,k)}|^2 \Bigg)\,,
\end{aligned} \tag{6.59}
$$

with $\mathcal{S}^{(\mathrm{s.o.})}_{ik}$ defined in eq. (4.26). Thus we see that the abelian term is the same as in eq. (4.24), while the non-abelian term in the strong-ordered limit is simplified a lot. We also see that the strongly-ordered doubly-soft limit is the same as the iterated doubly-soft limit recorded in eq. (6.13).

We remark that all the formulae in the strongly-ordered doubly-soft limit are symmetric in r and s, therefore, in the limit $\lambda_r \gg \lambda_s$, exactly the same formulae are valid.

Finally, we discuss the strongly-ordered soft-collinear limit, which is defined using a similar parametrization of the momenta as in section 4.3. The soft limit is defined by the rescaling $p_s = \lambda_s q_s$ with letting $\lambda_s \to 0$. The collinear limit is defined by the usual Sudakov parametrization of eq. (3.1), with rescaling of $k_{r\perp}$ as $k_{r\perp} = \lambda_r k_{r\perp}$ and letting $\lambda_r \to 0$. The limits λ_r, $\lambda_s \to 0$ are taken such that either $\lambda_s \gg \lambda_r$, or vice-versa. In the limit when $\lambda_r \gg \lambda_s$, the leading singular behaviour of the squared matrix element can be written as in

JHEP06(2005)024

eq. (6.2), which coincides with eq. (4.17). In the limit when $\lambda_s \gg \lambda_r$, the leading singular behaviour of the squared matrix element can be written as in eq. (6.36) which is also the same as eq. (4.17).

There is yet another strongly-ordered soft-collinear limit that occurs when a soft gluon of momentum $p_{(ir)} = \lambda_s q$ makes a collinear splitting into a quark-antiquark or gluon pair of momenta p_i and p_r. The collinear splitting is parametrized as in eq. (3.1) with rescaling of $k_{r\perp}$ as $k_{r\perp} = \lambda_r k_{r\perp}$. We consider the limit when $\lambda_s \to 0$ with $\lambda_s \gg \lambda_r$. When the splitting of the gluon is into a $q\bar{q}$ pair, we start with the factorization formula for soft-$q\bar{q}$ emission, eq. (4.20), suitably re-labelled,

$$\mathbf{S}_{ir}^{(q\bar{q})} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, \ldots)|^2 = \tag{6.60}$$
$$= (8\pi\alpha_\mathrm{s}\mu^{2\varepsilon})^2 \frac{1}{s_{ir}^2} \sum_{j\neq(ir)} \sum_{l\neq j,(ir)} \left(\frac{s_{ji}s_{lr} + s_{li}s_{jr} - s_{jl}s_{ir}}{s_{j(ir)}s_{l(ir)}} - 2\frac{s_{ji}s_{jr}}{s_{j(ir)}^2} \right) T_\mathrm{R}\, |\mathcal{M}_{m;(j,l)}^{(0)}(\ldots)|^2 \,.$$

The strongly-ordered limit can be obtained by substituting into eq. (6.60) the Sudakov parametrization of the momenta as in eq. (3.1) , which leads to

$$\begin{aligned} &\mathbf{S}_{(ir)} \mathbf{C}_{ir} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, \ldots)|^2 = \\ &\quad = (8\pi\alpha_\mathrm{s}\mu^{2\varepsilon})^2\, \delta_{f_i \bar{q}}\, \delta_{f_r q} \frac{1}{s_{ir}} \sum_{j,l\neq(ir)} \frac{1}{2} \mathcal{S}_{jl}^{\mu\nu}(ir)\, \langle\mu|\hat{P}_{\bar{q}q}^{(0)}|\nu\rangle |\mathcal{M}_{m;(j,l)}^{(0)}(\ldots)|^2 \,, \end{aligned} \tag{6.61}$$

i.e. to the soft limit of the collinear factorization formula (eq. (6.37)). When the splitting of the gluon is into a gluon pair, we start with the factorization formula for doubly-soft gluon emission, eq. (4.21), again suitably relabelled. The abelian part does not have a leading $\mathrm{O}[1/(\lambda_s^2\lambda_r^2)]$ singularity. Substituting the Sudakov parametrization of the momenta leads to an equation analogous to eq. (6.61) with the $\hat{P}_{\bar{q}q}^{(0)} \to \hat{P}_{gg}^{(0)}$ substitution.

7. Singly-unresolved limits of the factorization formulae for doubly-unresolved emission

We are now ready to list all the terms that are necessary for the explicit construction of the subtraction term $\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2$.

7.1 Singly-collinear limits of the doubly-unresolved factorization formulae

First we consider the limit of eq. (4.4) when partons i and r become collinear. In order to describe the $ir \to i + r$ collinear splitting, we introduce the Sudakov parametrization of the parton momenta as in eq. (6.45). The collinear region is identified by performing the rescaling $\kappa \to \lambda\kappa$ and studying the limit $\lambda \to 0$. This limit is the strongly-ordered doubly-collinear limit, therefore, the factorization formula $\mathbf{C}_{irs}|\mathcal{M}_{m+2}|^2$ factorizes as

$$\mathbf{C}_{ir}\mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \mathbf{C}_{(ir)s}\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 \,, \tag{7.1}$$

where the right-hand side is the iterated collinear limit of the collinear factorization formula given in eqs. (6.30)–(6.34).

JHEP06(2005)024

We parametrize the collinear limit for the other doubly-unresolved factorization formulae as in eq. (3.1). The factorization formula that is valid in the doubly-collinear limit (eq. (4.15)) is itself factorized in the two independent collinear splittings. Therefore, the limit of eq. (4.15) when parton i becomes collinear to r leaves that equation untouched, namely

$$\mathbf{C}_{ir}\mathbf{C}_{ir;js}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = \mathbf{C}_{ir}\mathbf{C}_{js}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2\,. \tag{7.2}$$

The same is true for eq. (4.17),

$$\mathbf{C}_{ir}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = \mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2\,. \tag{7.3}$$

The collinear limit of the factorization formula for the emission of a soft $q\bar{q}$-pair is clearly the strongly-ordered limit. The same is true for the non-abelian part of the two-gluon soft emission, thus,

$$\mathbf{C}_{ir}\mathbf{S}_{ir}^{(\mathrm{N})}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,\dots)|^2 = \mathbf{S}_{(ir)}\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,\dots)|^2\,, \tag{7.4}$$

where the right-hand side is the soft limit of the collinear-factorization formula given in eq. (6.37). The abelian part of the two-gluon soft emission cancels in $\mathbf{C}_{kt}\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$, therefore, we do not record it.

The computation of the remaining terms in eq. (5.10) is straightforward. The action of $\mathbf{C}_{ir}$ on eq. (4.32) reproduces the same,

$$\mathbf{C}_{ir}\mathbf{C}_{irs}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = \mathbf{C}_{irs}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2\,, \tag{7.5}$$

and so does it when applied on eq. (4.36),

$$\mathbf{C}_{ir}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2 = \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_j,p_s,\dots)|^2\,. \tag{7.6}$$

From eq. (4.37) we get

$$\begin{aligned}\mathbf{C}_{rs}\mathbf{C}_{irs}\mathbf{S}_{rs}^{(q\bar{q})}&|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = \\ &= (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\boldsymbol{T}_i^2\,T_{\mathrm{R}}\,\frac{2}{s_{rs}s_{i(rs)}}\left(\frac{z_i}{z_r+z_s}-\frac{s_{ik_{s\perp}}^2}{s_{rs}s_{i(rs)}}\right)|\mathcal{M}_m^{(0)}(p_i,\dots)|^2\,,\end{aligned} \tag{7.7}$$

and from eq. (4.41), with $z_r = 1 - z_s$, we find after some algebra,

$$\begin{aligned}\mathbf{C}_{rs}\mathbf{C}_{irs}\mathbf{S}_{rs}^{(\mathrm{nab})}&|\mathcal{M}_{m+2}^{(0)}(p_i,p_r,p_s,\dots)|^2 = \\ &= (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\times \\ &\times\boldsymbol{T}_i^2\,C_{\mathrm{A}}\,\frac{2}{s_{rs}s_{i(rs)}}\left[\frac{2z_i}{z_r+z_s}\left(\frac{z_r}{z_s}+\frac{z_s}{z_r}\right)+(1-\varepsilon)\frac{s_{ik_{s\perp}}^2}{s_{rs}s_{i(rs)}}\right]|\mathcal{M}_m^{(0)}(p_i,\dots)|^2\,.\end{aligned} \tag{7.8}$$

JHEP06(2005)024

7.2 Singly-soft limits of the doubly-unresolved factorization formulae

We consider the limit when the soft momentum is parametrized as $p_r = \lambda q_r$ and let $\lambda \to 0$ at fixed q_r. In this limit the triple-parton splitting kernels factorize into a soft function times the Altarelli-Parisi kernel obtained from the triple-parton kernel by deleting the soft parton. Therefore, we obtain

$$\begin{aligned} &\mathbf{S}_r \mathbf{C}_{irs} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \\ &\quad = 8\pi\alpha_s\mu^{2\varepsilon}\, P^{(\mathrm{S})}_{f_i f_r f_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon)\, \mathbf{C}_{is} |\mathcal{M}_{m+1}^{(0)}(p_i, p_s, \ldots)|^2 \,, \end{aligned} \tag{7.9}$$

with

$$P^{(\mathrm{S})}_{q_i \bar{q}'_r q'_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) = 0 \,, \tag{7.10}$$

$$P^{(\mathrm{S})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) = C_{\mathrm{F}} \frac{2}{s_{ir}} \frac{z_i}{z_r} + C_{\mathrm{A}} \left(\frac{s_{is}}{s_{ir}s_{rs}} + \frac{1}{s_{rs}} \frac{z_s}{z_r} - \frac{1}{s_{ir}} \frac{z_i}{z_r} \right), \tag{7.11}$$

$$P^{(\mathrm{S})}_{g_r q_s \bar{q}_i}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) = C_{\mathrm{F}} \frac{2 s_{is}}{s_{ir}s_{rs}} + C_{\mathrm{A}} \left(\frac{1}{s_{rs}} \frac{z_s}{z_r} + \frac{1}{s_{ir}} \frac{z_i}{z_r} - \frac{s_{is}}{s_{ir}s_{rs}} \right), \tag{7.12}$$

$$P^{(\mathrm{S})}_{g_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}; \varepsilon) = C_{\mathrm{A}} \left(\frac{s_{is}}{s_{ir}s_{rs}} + \frac{1}{s_{ir}} \frac{z_i}{z_r} + \frac{1}{s_{rs}} \frac{z_s}{z_r} \right). \tag{7.13}$$

The soft limit of the soft-collinear factorization formula in eq. (4.17) leaves it unchanged if $p_s \to 0$,

$$\mathbf{S}_s \mathbf{CS}_{ir;s} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \mathbf{CS}_{ir;s} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 \,. \tag{7.14}$$

However, it is crucial that in eq. (7.14) we use eq. (4.19). When only either parton r or s becomes soft, the asymptotic behaviour of the factorization formulae for the emission of a soft $\bar{q}q$ pair (in eq. (4.20)) or of a gluon pair (in eq. (4.21)) is

$$\mathbf{S}_r \mathbf{S}^{(q\bar{q})}_{rs} |\mathcal{M}_{m+2}^{(0)}(p_r, p_s, \ldots)|^2 = 0 \,, \tag{7.15}$$

$$\mathbf{S}_r \mathbf{S}^{(gg)}_{rs} |\mathcal{M}_{m+2}^{(0)}(p_r, p_s, \ldots)|^2 = \mathbf{S}_r \mathbf{S}_s |\mathcal{M}_{m+2}^{(0)}(p_r, p_s, \ldots)|^2 \,, \tag{7.16}$$

i.e., the iterated soft limit of eq. (6.13).

The computation of the remaining terms in eq. (5.9) is straightforward. From eq. (4.41) we get

$$\begin{aligned} &\mathbf{S}_s \mathbf{C}_{irs} \mathbf{S}^{(\mathrm{nab})}_{rs} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \\ &\quad = (8\pi\alpha_s\mu^{2\varepsilon})^2 \times \\ &\quad\quad \times \boldsymbol{T}_i^2\, C_{\mathrm{A}} \left(\frac{1}{s_{ir}s_{rs}} \frac{2z_i}{z_s} + \frac{1}{s_{is}s_{rs}} \frac{2z_i}{z_r} - \frac{1}{s_{ir}s_{is}} \frac{2z_i^2}{z_r z_s} \right) |\mathcal{M}_m^{(0)}(p_i, \ldots)|^2 \,. \end{aligned} \tag{7.17}$$

The action of $\mathbf{S}_s$ on eq. (4.34) leads to the same,

$$\mathbf{S}_s \mathbf{C}_{irs} \mathbf{CS}_{ir;s} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \mathbf{C}_{irs} \mathbf{CS}_{ir;s} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 \,, \tag{7.18}$$

and so does it when it is applied on eq. (4.45),

$$\mathbf{S}_s \mathbf{CS}_{ir;s} \mathbf{S}_{rs} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 = \mathbf{CS}_{ir;s} \mathbf{S}_{rs} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, p_s, \ldots)|^2 \,. \tag{7.19}$$

JHEP06(2005)024

Finally, we consider the limit when the soft momentum r is also collinear to the momentum of parton i. Clearly, in eq. (7.9) the operator $\mathbf{C}_{ir}$ acts only on the $P^{(\mathrm{S})}_{irs}$ functions,

$$\mathbf{C}_{ir}\mathbf{S}_r\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \\ = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\left[\mathbf{C}_{ir}P^{(\mathrm{S})}_{irs}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs};\varepsilon)\right]\mathbf{C}_{is}|\mathcal{M}^{(0)}_{m+1}(p_i,p_s,\ldots)|^2 . \quad (7.20)$$

In this limit, for the non-vanishing $P^{(\mathrm{S})}_{irs}$ functions we obtain

$$\mathbf{C}_{ir}P^{(\mathrm{S})}_{irs}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs};\varepsilon) = \boldsymbol{T}_i^2\,\frac{2}{s_{ir}}\,\frac{z_i}{z_r}\,. \quad (7.21)$$

The action of $\mathbf{C}_{js}$ on eq. (7.14) is

$$\mathbf{C}_{js}\mathbf{S}_s\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i,p_j,p_r,p_s,\ldots)|^2 = \\ = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{s_{ir}s_{js}}\,\frac{2z_j}{z_s}\,\boldsymbol{T}_j^2\,\langle\mathcal{M}^{(0)}_m(p_{ir},\ldots)|\,\hat{P}^{(0)}_{f_i,f_r}\,|\mathcal{M}^{(0)}_m(p_{ir},\ldots)\rangle\,, \quad (7.22)$$

while on eq. (7.16) it acts like

$$\mathbf{C}_{js}\mathbf{S}_r\mathbf{S}^{(gg)}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_j,p_r,p_s,\ldots)|^2 = \mathbf{S}_r\mathbf{C}_{js}\mathbf{S}_s|\mathcal{M}^{(0)}_{m+2}(p_j,p_r,p_s,\ldots)|^2\,, \quad (7.23)$$

which is given explicitly in eq. (6.17). Furthermore,[3]

$$\mathbf{C}_{rs}\mathbf{S}_s\mathbf{S}^{(\mathrm{nab})}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_r,p_s,\ldots)|^2 = \\ = -(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 C_{\mathrm{A}}\frac{1}{s_{rs}}\,\frac{2z_r}{z_s}\sum_j\sum_{l\neq j}\frac{1}{2}\,\mathcal{S}_{jl}(r)|\mathcal{M}^{(0)}_{m,(j,l)}(\ldots)|^2\,. \quad (7.24)$$

Applying the operator $\mathbf{C}_{js}\mathbf{S}_s$ on eq. (4.45) we get

$$\mathbf{C}_{js}\mathbf{S}_s\mathbf{CS}_{ir;s}\mathbf{S}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_j,p_r,p_s,\ldots)|^2 = \\ = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\,\boldsymbol{T}_i^2\,\frac{1}{s_{ir}}\,\frac{2z_i}{z_r}\,\boldsymbol{T}_j^2\,\frac{1}{s_{js}}\,\frac{2z_j}{z_s}\,|\mathcal{M}^{(0)}_m(p_i,p_j,\ldots)|^2\,, \quad (7.25)$$

which holds also for $j = i$. Finally, the action of $\mathbf{C}_{rs}$ on eq. (7.17) is

$$\mathbf{C}_{rs}\mathbf{S}_s\mathbf{C}_{irs}\mathbf{S}^{(\mathrm{nab})}_{rs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\boldsymbol{T}_i^2\,C_{\mathrm{A}}\,\frac{1}{s_{ir}s_{rs}}\,\frac{4z_i}{z_s}|\mathcal{M}^{(0)}_m(p_i,\ldots)|^2\,. \quad (7.26)$$

As we noted at the end of section 5, the factorization formulae of this section, with similar ones that are not listed, can be used to write the terms on the right-hand side of eq. (5.8) in the form given in eqs. (5.9)–(5.11).

[3]In the $\mathbf{C}_{rs}\mathbf{S}_s$ limit the momentum p_r is the correct variable in the factorized matrix element (see eq. (3.16)).

8. The singly- and doubly-unresolved limits of the NNLO subtraction term

We now turn to the proof of eqs. (5.2)–(5.7). The explicit check of these equations involves straightforward, but lengthy algebra. Therefore, we only outline the proof, which uses the fact that $\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2 = \mathbf{A}_1\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2$, and let the interested readers go through the details.

Let us first consider the singly-unresolved limits. Grouping the terms as in eqs. (3.21) and (3.22), we obtain

$$\boldsymbol{X}_s\mathbf{A}_1\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \left(\boldsymbol{X}_s\mathbf{A}_2 + (\boldsymbol{X}_s\mathbf{A}_1 - \boldsymbol{X}_s)\mathbf{A}_2\right)|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{8.1}$$

where $\boldsymbol{X}_s$ stands for either the collinear limit $\mathbf{C}_{js}$, or the soft one $\mathbf{S}_s$. As a result,

$$\begin{aligned}\boldsymbol{X}_s(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_1\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 &= \boldsymbol{X}_s|\mathcal{M}_{m+2}^{(0)}|^2 + \\ &\quad + (\boldsymbol{X}_s\mathbf{A}_1 - \boldsymbol{X}_s)(1 - \mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2\,.\end{aligned} \tag{8.2}$$

The factor in the first parenthesis on the right-hand side contains operators that take the iterated singly-unresolved limits, which give vanishing contributions on $(1 - \mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2$,

$$(\boldsymbol{X}_s\mathbf{A}_1 - \boldsymbol{X}_s)(1 - \mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,. \tag{8.3}$$

We checked eq. (8.3) explicitly for all the terms in eqs. (3.21) and (3.22). It is a consequence of eqs. (4.59)–(4.62), which hold also in the strongly-ordered doubly-unresolved (or equivalently in the iterated singly-unresolved) regions. Thus only the first term on the right-hand side of eq. (8.2) remains, and that is the relevant singly-unresolved limit of the squared matrix element. Therefore, eqs. (5.2) and (5.3) hold.

Turning to the doubly-unresolved limits, we first observe that according to eqs. (4.59)–(4.62) we have

$$\boldsymbol{Y}_{rs}\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \boldsymbol{Y}_{rs}|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{8.4}$$

where $\boldsymbol{Y}_{rs}$ stands for any of the operators $\mathbf{C}_{irs}$, $\mathbf{C}_{ir;js}$, $\mathbf{CS}_{ir;s}$ and $\mathbf{S}_{rs}$. Thus, all we have to prove is the following relation:

$$\boldsymbol{Y}_{rs}(\mathbf{A}_1 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,. \tag{8.5}$$

We checked the validity of eq. (8.5) explicitly for the four different limits. In general it follows from

$$\boldsymbol{Y}_{rs}(\mathbf{A}_1 - \mathbf{A}_{12})|\mathcal{M}_{m+2}^{(0)}|^2 = \boldsymbol{Y}_{rs}\mathbf{A}_1(1 - \mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2\,, \tag{8.6}$$

and observing that in any term $\boldsymbol{X}_s|\mathcal{M}_{m+2}^{(0)}|^2$ in the NLO subtraction $\mathbf{A}_1|\mathcal{M}_{m+2}^{0)}|^2$ the dependence on s is already factorized, therefore, the terms $\boldsymbol{Y}_{rs}\boldsymbol{X}_s|\mathcal{M}_{m+2}^{(0)}|^2$ are factorization formulae valid in the iterated singly-unresolved limit, or according to the results of section 6.5, in the strongly-ordered doubly-unresolved regions. In these regions eqs. (4.59)–(4.62) hold, therefore,

$$\boldsymbol{Y}_{rs}\mathbf{A}_1(1 - \mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2 = 0\,, \tag{8.7}$$

which leads to eq. (8.5).

JHEP06(2005)024

It is worth noting that in eqs. (8.3) and (8.5) we have to take unresolved limits of factorization formulae that are valid in different unresolved regions of the phase space. For instance, eq. (8.3) contains terms such as $\mathbf{S}_s\mathbf{C}_{sr}\mathbf{S}_r\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2$. According to eq. (5.11) such terms contain the factorization formula $\mathbf{C}_{sr}\mathbf{S}_r\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}|^2$, given explicitly by

$$\begin{aligned}
&\mathbf{C}_{sr}\mathbf{S}_r\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \times \\
&\quad \times \frac{2}{s_{rs}}\frac{z_s}{z_r}\boldsymbol{T}^2_s\frac{1}{s_{is}}\langle\mathcal{M}^{(0)}_m(p_{is},\ldots)|\hat{P}^{(0)}_{f_i,f_s}(z_i,k_{s\perp};\varepsilon)|\mathcal{M}^{(0)}_m(p_{is},\ldots)\rangle\,.
\end{aligned} \tag{8.8}$$

In the region where $p_s \to 0$, this expression contains the ambiguous limit $\lim_{z_s\to 0} z_s/z_s$. This limit can be computed using the constraints the momentum fractions have to obey. The z_s in the numerator is relevant in the collinear limit of partons s and r, where it fulfills the constraint $z_s + z_r = 1$. The z_s in the denominator is relevant in the region where the partons i and s are collinear when $z_s + z_i = 1$. Thus the momentum fractions, needed to define the various collinear limits, have to be extended to other parts of the phase space such that they fulfill different constraints in different regions. For instance, using the energy fractions, mentioned at the end of section 4.1, we find that the soft limit of eq. (8.8) is

$$\mathbf{S}_s\mathbf{C}_{sr}\mathbf{S}_r\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s,\ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{2}{s_{rs}s_{is}}\frac{z_i}{z_r}C_{\mathrm{A}}\,\boldsymbol{T}^2_i|\mathcal{M}^{(0)}_m(p_i,\ldots)|^2\,. \tag{8.9}$$

The cancellations discussed in eqs. (8.3) and (8.5) all take place with momentum fractions defined as energy fractions.

9. A possible general subtraction method for computing NNLO corrections to jet cross sections

Building on our studies in the previous sections concerning the matching of the various singly- and doubly-unresolved limits of the squared matrix element, in this section we outline a possible general subtraction method for computing NNLO corrections to jet cross sections.

The full cross section to NNLO accuracy can schematically be written as

$$\sigma_{\mathrm{NNLO}} = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}}\,. \tag{9.1}$$

Assuming an m-jet quantity, the leading-order contribution is the integral of the fully differential Born cross section $\mathrm{d}\sigma^{\mathrm{B}}_m$ of m final-state partons over the available m-parton phase space defined by the jet function J_m,

$$\sigma^{\mathrm{LO}} = \int_m \mathrm{d}\sigma^{\mathrm{B}}_m J_m\,. \tag{9.2}$$

The NLO contribution is a sum of two terms, the real and virtual corrections,

$$\sigma^{\mathrm{NLO}} = \int_{m+1}\mathrm{d}\sigma^{\mathrm{R}}_{m+1}J_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{V}}_m J_m\,. \tag{9.3}$$

JHEP06(2005)024

Here the notation for the integrals indicate that the real correction involves $m+1$ final-state partons, one of those being unresolved, while the virtual correction has m-parton kinematics. The NNLO correction is a sum of three contributions, the double-real, the one-loop single-unresolved real-virtual and the two-loop double-virtual terms,

$$\sigma^{\rm NNLO} = \int_{m+2} {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} + \int_{m+1} {\rm d}\sigma^{\rm RV}_{m+1} J_{m+1} + \int_{m} {\rm d}\sigma^{\rm VV}_{m} J_{m} \,. \tag{9.4}$$

Here the notation for the integrals indicate that the double-real corrections involve $m+2$ final-state partons, the real-virtual contribution involves $m+1$ final-state partons and the double virtual term is an integral over the phase space of m partons, and the phase spaces are restricted by the corresponding jet functions J_n that define the physical quantity.

The various contributions to the NLO and NNLO corrections are divergent. We regularize these singularities using dimensional regularization, that is defining the integrals in $d = 4 - 2\varepsilon$ dimiensions. After ultraviolet (UV) renormalization of the one- and two-loop matrix elements involved in ${\rm d}\sigma^{\rm V}$, ${\rm d}\sigma^{\rm RV}$ and ${\rm d}\sigma^{\rm VV}$ all the contributions to $\sigma_{\rm NNLO}$ are UV finite. In general, the jet function of $m+n$ partons vanishes if $n+1$ or more partons become unresolved. Explicitly, J_m vanishes when one parton becomes soft or collinear to another one,

$$J_m(p_1, \dots, p_m) \to 0 \,, \qquad \text{if} \qquad p_i \cdot p_j \to 0 \,, \tag{9.5}$$

and J_{m+1} vanishes when two partons become simultaneously soft and/or collinear,

$$J_{m+1}(p_1, \dots, p_{m+1}) \to 0 \,, \qquad \text{if} \qquad p_i \cdot p_j \text{ and } p_k \cdot p_l \to 0 \quad (i \neq k) \,. \tag{9.6}$$

Therefore, the Born contribution ${\rm d}\sigma^{\rm B}_m$ is integrable over the one- or two-parton infrared (IR) regions of the phase space. Also the real and virtual contributions ${\rm d}\sigma^{\rm R}_{m+1} J_{m+1}$ and ${\rm d}\sigma^{\rm V}_m J_m$ vanish when we integrate over the two-parton IR region of the phase space, but are separately divergent in $d = 4$ dimensions when only one (real or virtual) parton becomes soft or collinear to another parton. Their sum is finite for *infrared-safe* observables. Formally, infrared safety is expressed by the following requirements for the jet function,

$$J_{n+1}(p_1, \dots, p_j = \lambda q, \dots, p_{n+1}) \to J_n(p_1, \dots, p_{n+1}) \quad \text{if} \quad \lambda \to 0 \,, \tag{9.7}$$

$$J_{n+1}(p_1, \dots, p_i, \dots, p_j, \dots, p_{n+1}) \to J_n(p_1, \dots, p, \dots, p_{n+1}) \quad \text{if} \quad p_i \to zp \,,\ p_j \to (1-z)p \tag{9.8}$$

for all $n \geq m$. Equations (9.7) and (9.8) respectively guarantee that the jet observable is infrared and collinear safe for *any* number n of final-state partons, i.e. to *any* order in QCD perturbation theory if there is only one infrared parton. The n-parton jet function J_n on the right-hand side of eq. (9.7) is obtained from the original J_{n+1} by removing the soft parton p_j, and that on the right-hand side of eq. (9.8) by replacing the collinear partons $\{p_i, p_j\}$ by $p = p_i + p_j$. For infrared-safe observables, process- and observable-independent techniques exist to cancel the IR divergences by devising an approximate fully differential cross section over the m-parton phase space, ${\rm d}\sigma^{\rm R,A}_{m+1}$ such that (i) ${\rm d}\sigma^{\rm R,A}_{m+1}$ matches the pointwise singular behaviour of ${\rm d}\sigma^{\rm R}_{m+1}$ in the one-parton IR regions of the phase space in any dimensions (ii) and it can be integrated analytically over the one-parton phase space, so we can combine

JHEP06(2005)024

it with $\mathrm{d}\sigma_m^{\mathrm{V}}$ before integration. We then write

$$\sigma^{\mathrm{NLO}} = \int_{m+1} \left[\mathrm{d}\sigma_{m+1}^{\mathrm{R}} J_{m+1} - \mathrm{d}\sigma_{m+1}^{\mathrm{R,A}} J_m\right] + \int_m \left[\mathrm{d}\sigma_m^{\mathrm{V}} + \int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{R,A}}\right] J_m\,, \tag{9.9}$$

where both integrals on the right-hand side are finite in $d = 4$ dimensions (in order to prove the finiteness of the first integral, we have to use eqs. (9.7) and (9.8)). The final result is that we were able to rewrite the two NLO contributions in eq. (9.3) as a sum of two finite integrals,

$$\sigma^{\mathrm{NLO}} = \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{NLO}} + \int_m \mathrm{d}\sigma_m^{\mathrm{NLO}}\,, \tag{9.10}$$

that are integrable in four dimensions using standard numerical techniques.

The three contributions to the NNLO corrections in eq. (9.4) are separately divergent in both the one- and the two-parton IR regions, but their sum is finite for infrared-safe observables. To NNLO accuracy, the formal requirement of infrared safety is expressed by the following relations,

$$J_{n+2}(p_1,\dots,p_j=\lambda q,\dots,p_k=\lambda r,\dots,p_{n+2}) \to J_n(p_1,\dots,p_{n+2}) \quad \text{if} \quad \lambda\to 0\,, \tag{9.11}$$

$$\begin{aligned} &J_{n+2}(p_1,\dots,p_j=\lambda q,\dots,p_k,\dots,p_l,\dots,p_{n+2}) \to J_n(p_1,\dots,p_{kl},\dots,p_{n+2}) \\ &\qquad \text{if} \quad \lambda\to 0 \quad \text{and} \quad p_k\to z_k p_{kl}\,,\ p_l\to(1-z_k)p_{kl}\,, \end{aligned} \tag{9.12}$$

$$\begin{aligned} &J_{n+2}(p_1,\dots,p_i,\dots,p_j,\dots,p_k,\dots,p_l,\dots,p_{n+2}) \to J_n(p_1,\dots,p_{ij},\dots,p_{kl},\dots,p_{n+2}) \\ &\qquad \text{if} \quad p_i\to z_i p_{ij}\,,\ p_j\to(1-z_i)p_{ij} \quad \text{and} \quad p_k\to z_k p_{kl}\,,\ p_l\to(1-z_k)p_{kl}\,, \end{aligned} \tag{9.13}$$

$$\begin{aligned} &J_{n+2}(p_1,\dots,p_i,\dots,p_j,\dots,p_k,\dots,p_{n+2}) \to J_n(p_1,\dots,p_{ijk},\dots,p_{n+2}) \\ &\qquad \text{if} \quad p_i\to z_i p_{ijk}\,, \quad p_j\to z_j p_{ijk} \quad \text{and} \quad p_k\to(1-z_i-z_j)p_{ijk} \end{aligned} \tag{9.14}$$

for all $n \geq m$ (as implicitly mentioned before, J_{m+2} vanishes if three or more partons become unresolved). Eqs. (9.11)–(9.14) guarantee that the jet observable is infrared safe for *any* number n of final-state partons, i.e. to *any* order in QCD perturbation theory if there are exactly two unresolved partons. The n-parton jet function J_n on the right-hand side of eq. (9.11) is obtained from the original J_{n+2} by removing the soft partons p_j and p_k, that on the right-hand side of eq. (9.12) by removing the soft parton p_j and replacing the collinear partons $\{p_k, p_l\}$ by $p_{kl} = p_k + p_l$, that on the right-hand side of eq. (9.13) by replacing the collinear partons $\{p_i, p_j\}$ by $p_{ij} = p_i + p_j$, and $\{p_k, p_l\}$ by $p_{kl} = p_k + p_l$ and that on the right-hand side of eq. (9.14) by replacing the collinear partons $\{p_i, p_j, p_k\}$ by $p_{ijk} = p_i + p_j + p_k$.

In order to cancel the one- and two-parton singularities between the double-real, real-virtual and double-virtual terms in eq. (9.4), we proceed like in eq. (9.9), namely, we introduce approximate cross sections $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$ and $\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}}$ and rewrite eq. (9.4) as

$$\begin{aligned} \sigma^{\mathrm{NNLO}} = &\int_{m+2} \left[\mathrm{d}\sigma^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} J_m\right] + \int_{m+1} \left[\mathrm{d}\sigma^{\mathrm{RV}} J_{m+1} - \mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} J_m\right] + \\ &+ \int_m \mathrm{d}\sigma^{\mathrm{VV}} J_m + \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} J_m + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} J_m\,. \end{aligned} \tag{9.15}$$

JHEP06(2005)024

The fully differential approximate cross section to the double-real emission is constructed such that it has the same pointwise singular behaviour in d dimensions as $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$ in the two-parton infrared regions (hence the subscript 2 on A in the superscript). Then eqs. (9.11)–(9.14) guarantee that the $(m+2)$-parton integral in eq. (9.15) is finite over the doubly-unresolved regions of the phase space. However, this integral is still divergent in the singly-unresolved regions, because both the fully differential cross sections $\mathrm{d}\sigma^{\mathrm{RR}}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ as well as the jet functions J_{m+2} and J_m have different limits in the singly-unresolved regions.

The real-virtual contribution has two types of singularities: (i) explicit ε poles in the loop amplitude and (ii) singular behaviour in the singly-unresolved regions of the phase space. To regularize the latter, the fully differential approximate cross section, $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$, to the real-virtual contribution must have the same pointwise singular behaviour in d dimensions as $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ itself in the one-parton infrared regions (hence the subscript 1 on A in the superscript). Using such a regulator, the difference $\left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} J_m\right]$ is still not integrable over the full $(m+1)$-parton phase space in $d = 4$ dimensions, simply because the integrand still contains ε poles in phase space regions away from the one-parton unresolved infrared regions. The pole structures of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ and $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ are different.

In order to cancel the remaining singularities in the $(m+2)$-parton and $(m+1)$-parton integrals in the first line of eq. (9.15), we perform a subtraction procedure between these two integrals, which is very similar to that used in a NLO computation, eq. (9.9). This amounts to devising fully differential approximate cross sections to both $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ that match the singular behaviour of these cross sections in the singly-unresolved infrared regions in d dimensions, which are denoted by $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$, respectively. Thus, the full NNLO cross section is written as

$$
\begin{aligned}
\sigma^{\mathrm{NNLO}} = & \int_{m+2}\left[\mathrm{d}\sigma^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} + \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m\right] + \\
& + \int_{m+1}\left[\mathrm{d}\sigma^{\mathrm{RV}} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} J_m\right] + \int_{m+2}\left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m\right] + \\
& + \int_m \mathrm{d}\sigma^{\mathrm{VV}} J_m + \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} J_m \,. \qquad (9.16)
\end{aligned}
$$

The first integral on the right-hand side of this equation is finite in $d = 4$ dimensions by construction. Hence, this first integral can be performed numerically in four dimensions using standard Monte Carlo techniques.

The singularities are associated to the integrals in the second and third lines of eq. (9.16). We shall show in a separate publication that the integration of $[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m]$ can be performed analytically over the one-parton phase space of the unresolved parton of the $(m+2)$-parton phase space, leading to ε poles, which cancel exactly the remaining ε poles in the integrand $[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} J_m]$. As a result the integrals in the second line of eq. (9.16) can be combined into a single $(m+1)$-parton integral, where the integrand is free of ε poles and can be integrated numerically in four dimensions. We emphasize that this statement does not follow from unitarity, but has to be proven.

JHEP06(2005)024

Once we have shown the finiteness of the combination of integrals in the second line of eq. (9.16), it follows from the Kinoshita-Lee-Nauenberg theorem that the combination of the integrals in the third line must be finite, though separately all three are divergent. If one is able to carry out analytically the integration of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ over the two-parton subspace and that of $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$ over the one-parton subspace of the doubly- and singly-unresolved regions of the $(m+2)$-parton and $(m+1)$-parton phase spaces, respectively, leading to ε poles, we can combine these poles with those in the double-virtual contribution. As a result all the divergences cancel and the remaining m-parton integral can be performed numerically in four dimensions.

The final result of the above manipulations is that we rewrite eq. (9.4) as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} + \int_{m} \mathrm{d}\sigma_{m}^{\mathrm{NNLO}} \,, \tag{9.17}$$

that is, a sum of three integrals,

$$\mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} = \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} J_m - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} + \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} J_m\right]_{\varepsilon=0} \,, \tag{9.18}$$

$$\mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} = \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} J_{m+1} - \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} J_m + \int_1 \left(\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} J_m\right)\right]_{\varepsilon=0} \,, \tag{9.19}$$

and

$$\mathrm{d}\sigma_{m}^{\mathrm{NNLO}} = \left[\mathrm{d}\sigma^{\mathrm{VV}} + \int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} + \int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}\right]_{\varepsilon=0} J_m \,, \tag{9.20}$$

each integrable in four dimensions using standard numerical techniques.

The factorization formulae presented in the previous sections can be used in constructing the approximate cross sections $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$, $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ and $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$. However, we emphasize that the momenta in the matrix elements on the right hand sides of the factorization formulae are unambigously defined only in the strict unresolved limits. Therefore the factorization formulae cannot be directly used as true subtraction formulae, unless we can explicitly specify which partons become unresolved — either by partitioning the squared matrix element as in ref. [38], or using the measurement function as in ref. [3], or by partitioning the phase space as in ref. [4]. These procedures lead to the so-called residue methods which are well understood in NLO computations. If one wishes to avoid specifying the unresolved partons, then one has to implement exact factorization of the phase space, such that the integrations over the unresolved momenta can be carried out and momentum conservation is maintained as done in the case of dipole subtraction method of ref. [6], which is worked out completely for NLO computations.

10. Conclusions

In this paper we studied the infrared structure of the known factorization formulae for tree-level QCD squared matrix elements in all the possible soft and collinear limits. We presented new factorization formulae for the colour-correlated and the spin-correlated squared matrix elements that we termed iterated singly-unresolved limits. We pointed out that soft factorization formulae do not exist for the simultaneously spin- and colour-correlated squared matrix elements which indicates that in the general subtraction scheme envis-

JHEP06(2005)024

aged by us, the azimuthally correlated singly-collinear subtraction terms must not contain colour correlations. This can be achieved naturally for those processes when the colour charges in the colour-correlated squared matrix elements can be factorized completely (see appendix A). We derived the factorization formulae of the squared matrix element that are valid in the strongly-ordered doubly-unresolved regions of the phase space and showed that these are equal to the iterated singly-unresolved factorization formulae. We dealt only with final state singularities. Using crossing symmetry, it is possible to extend our results to cases when initial state partons become unresolved, which we shall present in a separate publication.

We introduced a new formal notation for denoting the various limits in the overlapping unresolved parts of the phase space. Using this notation, we explicitly constructed subtraction terms that regularize the kinematical singularities of the squared matrix element in all singly- and doubly-unresolved parts of the phase space and demonstrated that the subtraction terms avoid all possible double and triple subtractions. The relevant subtraction terms were presented in sections 4 and 5. As a result the regularized squared matrix element is integrable over all phase space regions where at most two partons become unresolved. We emphasize however, that the subtraction terms derived in this paper can be used as true subtraction terms only if the unresolved momenta are specified, like in refs. [3, 4] in the case of NLO computations. In an approach, where any parton can become unresolved, the factorization formulae have to be extended over the whole phase space. This extension requires a phase-space factorization that maintains momentum conservation exactly, as done in ref. [6] in the case of NLO computations, but such that in addition it respects the structure of the delicate cancellations among the various subtraction terms. So far none of these methods has been demonstrated to be applicable in practical computations of NNLO corrections. The kinematics of two-jet production in electron-positron annihilation is sufficiently simple so that we were able to implement the subtraction scheme presented in this paper for the case of $m = 2$. We computed the cross sections in eqs. (9.18) and (9.19) and found numerically stable results for any observable. The details of the computations will be given elsewhere.

Acknowledgments

Z.T. thanks the INFN, Sez. di Torino, V.D.D. thanks the Nucl. Res. Inst. of the HAS for their kind hospitality during the long course of this work. We are grateful to S. Catani for his comments on the manuscript. This research was supported in part by the Hungarian Scientific Research Fund grant OTKA T-038240.

A. Explicit computation of the soft limits of known colour-correlated squared matrix elements

In this appendix we check the validity of the soft factorization formulae eqs. (6.8) and (6.11) using the colour-correlated squared matrix elements for the $e^+e^- \to q\bar{q} + ng$ processes ($n = 1, 2$).

JHEP06(2005)024

A.1 Soft limit of $|\mathcal{M}^{(0)}_{3;(i,k)}|^2$

Using colour conservation, we can compute the product of the colour charges acting on the three-parton colour space in terms of Casimir invariants [6],

$$\boldsymbol{T}_q \cdot \boldsymbol{T}_{\bar{q}} = \frac{C_{\rm A}}{2} - C_{\rm F}\,, \qquad \boldsymbol{T}_g \cdot \boldsymbol{T}_q = \boldsymbol{T}_g \cdot \boldsymbol{T}_{\bar{q}} = -\frac{C_{\rm A}}{2}\,. \tag{A.1}$$

Thus the colour charges in the colour-correlated squared matrix elements for the $e^+e^- \to q\bar{q}g$ process can be factorized completely [38]. If the quark, antiquark and gluon is labelled (in this order) 1, 2 and 3, then we have

$$|\mathcal{M}^{(0)}_{3;(1,2)}|^2 = \left(\frac{C_{\rm A}}{2} - C_{\rm F}\right) |\mathcal{M}^{(0)}_3|^2\,, \tag{A.2}$$

and

$$|\mathcal{M}^{(0)}_{3;(1,3)}|^2 = |\mathcal{M}^{(0)}_{3;(2,3)}|^2 = -\frac{C_{\rm A}}{2} |\mathcal{M}^{(0)}_3|^2\,. \tag{A.3}$$

Consequently, the limit of the colour-correlated squared matrix elements when p_3 gets soft is simply colour factors times the soft limit of the squared matrix element [39],

$$|\mathcal{M}^{(0)}_{3;(1,2)}(p_1,p_2,p_3)|^2 \simeq -8\pi\alpha_{\rm s}\mu^{2\varepsilon} \left(\frac{C_{\rm A}}{2} - C_{\rm F}\right) \mathcal{S}_{12}(3) |\mathcal{M}^{(0)}_{2;(1,2)}(p_1,p_2)|^2\,, \tag{A.4}$$

which — using $\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 |\mathcal{M}^{(0)}_2(p_1,p_2)\rangle = -C_{\rm F} |\mathcal{M}^{(0)}_2(p_1,p_2)\rangle$ — can also be obtained from eq. (6.8), and

$$|\mathcal{M}^{(0)}_{3;(1,3)}(p_1,p_2,p_3)|^2 \simeq 8\pi\alpha_{\rm s}\mu^{2\varepsilon} \frac{C_{\rm A}}{2} \mathcal{S}_{12}(3) |\mathcal{M}^{(0)}_{2;(1,2)}(p_1,p_2)|^2\,, \tag{A.5}$$

which can also be obtained from eq. (6.11).

A.2 Soft limit of $|\mathcal{M}^{(0)}_{4;(i,k)}|^2$

The colour-correlated squared matrix elements for the $e^+e^- \to q_1\bar{q}_2g_3g_4$ process were computed in appendix B of ref. [40].[4] We rewrite the expressions presented there using our notation:

$$\left|\mathcal{M}^{(0)}_{4;(i,k)}\right|^2 = -\frac{1}{2} N_{\rm c} C_{\rm F} \left(C_{\rm F}^2 M_0^{ik} + C_{\rm A} C_{\rm F} M_x^{ik} + C_{\rm A}^2 M_{xx}^{ik}\right)\,, \tag{A.6}$$

where the non-vanishing elements of the matrices M_0^{ik}, M_x^{ik}, M_{xx}^{ik} are given by

$$M_0^{12} = 2|m_3|^2\,, \tag{A.7}$$

$$M_x^{12} = -3|m_3|^2 + |m_1|^2 + |m_2|^2\,, \qquad M_x^{13} = M_x^{14} = M_x^{23} = M_x^{24} = |m_3|^2\,, \tag{A.8}$$

$$M_{xx}^{34} = \frac{1}{2}(|m_1|^2 + |m_2|^2)\,, \qquad M_{xx}^{12} = |m_3|^2 - M_{xx}^{34}\,, \tag{A.9}$$

$$M_{xx}^{13} = M_{xx}^{24} = -\frac{1}{2}(|m_3|^2 - |m_1|^2)\,, \quad M_{xx}^{14} = M_{xx}^{23} = -\frac{1}{2}(|m_3|^2 - |m_2|^2)\,, \tag{A.10}$$

[4] Note a typographic error in eqs. (B11)–(B13) of ref. [40]: the labelling convention given in that paper is correct for the process $e^+e^- \to q_1\bar{q}_2g_3g_4$ we consider here. In order to get the M_0^{ik}, M_x^{ik} and M_{xx}^{ik} matrices relevant to the labelling convention used in ref. [40] the 2, 3 and 4 indices of the M_0^{ik} matrices should be cyclicly permuted, (2,3,4) → (4,2,3).

JHEP06(2005)024

and the $|m_i|^2$ functions are the helicity-summed squared helicity amplitudes with all coupling factors included. In terms of the amplitudes defined in appendix A of ref. [40]

$$|m_1|^2 = \sum_{\text{hel}} |m(1,3,4,2)|^2 , \qquad |m_2|^2 = \sum_{\text{hel}} |m(1,4,3,2)|^2 , \tag{A.11}$$

and $m_3 = m_1 + m_2$. Using the soft factorization formula for the helicity amplitudes, it is not difficult to obtain the following factorization formulae for the $|m_i|^2$ functions when $p_4 \to 0$:

$$|m_1(p_1,p_2,p_3,p_4)|^2 \simeq 8\pi\alpha_{\rm s}\mu^{2\varepsilon}\mathcal{S}_{23}(4)\frac{1}{N_{\rm c}C_{\rm F}}|\mathcal{M}_3^{(0)}(p_1,p_2,p_3)|^2 , \tag{A.12}$$

$$|m_2(p_1,p_2,p_3,p_4)|^2 \simeq 8\pi\alpha_{\rm s}\mu^{2\varepsilon}\mathcal{S}_{13}(4)\frac{1}{N_{\rm c}C_{\rm F}}|\mathcal{M}_3^{(0)}(p_1,p_2,p_3)|^2 , \tag{A.13}$$

$$|m_3(p_1,p_2,p_3,p_4)|^2 \simeq 8\pi\alpha_{\rm s}\mu^{2\varepsilon}\mathcal{S}_{12}(4)\frac{1}{N_{\rm c}C_{\rm F}}|\mathcal{M}_3^{(0)}(p_1,p_2,p_3)|^2 . \tag{A.14}$$

Substituting eqs. (A.7)–(A.14) into eq. (A.6), we obtain the soft factorization formulae for the colour-correlated squared matrix elements, when $p_4 \to 0$:

$$\left|\mathcal{M}_{4;(i,k)}^{(0)}(p_1,p_2,p_3,p_4)\right|^2 \simeq -8\pi\alpha_{\rm s}\mu^{2\varepsilon}\left|\mathcal{M}_3^{(0)}(p_1,p_2,p_3)\right|^2 \mathcal{F}_{ik}(p_1,p_2,p_3,p_4) , \tag{A.15}$$

where

$$\mathcal{F}_{12}(p_1,p_2,p_3,p_4) = \left(C_{\rm F} - \frac{C_{\rm A}}{2}\right)\left[(C_{\rm F}-C_{\rm A})\mathcal{S}_{12}(4) + \frac{C_{\rm A}}{2}\left(\mathcal{S}_{13}(4)+\mathcal{S}_{23}(4)\right)\right] , \tag{A.16}$$

$$\mathcal{F}_{13}(p_1,p_2,p_3,p_4) = \frac{C_{\rm A}}{2}\left[\left(C_{\rm F} - \frac{C_{\rm A}}{2}\right)\mathcal{S}_{12}(4) + \frac{C_{\rm A}}{2}\mathcal{S}_{23}(4)\right] , \tag{A.17}$$

$$\mathcal{F}_{14}(p_1,p_2,p_3,p_4) = \frac{C_{\rm A}}{2}\left[\left(C_{\rm F} - \frac{C_{\rm A}}{2}\right)\mathcal{S}_{12}(4) + \frac{C_{\rm A}}{2}\mathcal{S}_{13}(4)\right] , \tag{A.18}$$

$$\mathcal{F}_{34}(p_1,p_2,p_3,p_4) = \left(\frac{C_{\rm A}}{2}\right)^2 \left[\mathcal{S}_{13}(4)+\mathcal{S}_{23}(4)\right] . \tag{A.19}$$

Using eq. (A.1) it is straightforward to check that the results in eq. (A.15), eqs. (A.16) and (A.17) can also be obtained from eq. (6.8), and those in eq. (A.15), (A.18) and (A.19) can also be obtained from eq. (6.11).

B. Collinear limit of the spin-polarisation tensor for the process $e^+e^- \to q\bar{q}g$

Consider the *tree-level* matrix element for the production of a quark-antiquark pair and a gluon in electron-positron annihilation,

$$\mathcal{M}_{q_1,\bar{q}_2,g_3}^{c_1,c_2,c_3;s_1,s_2,\mu_3}(p_1,p_2,p) = \langle c_1c_2c_3| \otimes \langle s_1s_2\mu_3||\mathcal{M}_3^{(0)}(p_1,p_2,p)\rangle , \tag{B.1}$$

where $\{c_1,c_2,c_3\}$, $\{s_1,s_2,\mu_3\}$ and $\{q_1,\bar{q}_2,g_3\}$ are the colour, spin and flavour indices of the partons. We define the spin-polarization tensor as in eq. (6.20):

$$\mathcal{T}_{q\bar{q}g}^{\mu\nu}(p_1,p_2,p) = \langle\mathcal{M}_3^{(0)}(p_1,p_2,p)|\mu\rangle\langle\nu|\mathcal{M}_3^{(0)}(p_1,p_2,p)\rangle , \tag{B.2}$$

JHEP06(2005)024

so $-g_{\mu\nu}\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = |\mathcal{M}^{(0)}_3(p_1,p_2,p)|^2$.

In this appendix, we examine the collinear limit of $\langle\mu|\hat{P}^{(0)}_{gg}(z,k_\perp;\varepsilon)|\nu\rangle\,\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p)$ (we suppress the second argument of the Altarelli-Parisi kernel, which equals $1-z$). From appendix D of ref. [6] we have (in $d=4$ dimensions)

$$\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = -\frac{1}{y_{1p}^2+y_{2p}^2+2y_{12}}\,|\mathcal{M}^{(0)}_3(p_1,p_2,p)|^2 T^{\mu\nu}, \tag{B.3}$$

where $y_{ij} = s_{ij}/Q^2 \equiv s_{ij}/(p_1+p_2+p)^2$ and

$$\begin{aligned} T^{\mu\nu} = {} & +2\frac{p_1^\mu p_2^\nu}{Q^2} + 2\frac{p_2^\mu p_1^\nu}{Q^2} - 2\frac{y_{2p}}{y_{1p}}\frac{p_1^\mu p_1^\nu}{Q^2} - 2\frac{y_{1p}}{y_{2p}}\frac{p_2^\mu p_2^\nu}{Q^2} + \\ & + \frac{y_{12}-(y_{12}+y_{2p})^2}{y_{1p}}\left[\frac{p_1^\mu p^\nu}{Q^2} + \frac{p^\mu p_1^\nu}{Q^2}\right] + \frac{y_{12}-(y_{12}+y_{1p})^2}{y_{2p}}\left[\frac{p_2^\mu p^\nu}{Q^2} + \frac{p^\mu p_2^\nu}{Q^2}\right] + \\ & + \frac{1}{2}\left(y_{1p}^2+y_{2p}^2\right) g^{\mu\nu}\,. \end{aligned} \tag{B.4}$$

We also have (in $d=4$ dimensions)

$$|\mathcal{M}^{(0)}_3(p_1,p_2,p)|^2 = C_{\mathrm{F}}\,\frac{8\pi\alpha_{\mathrm{s}}}{Q^2}\,\frac{y_{1p}^2+y_{2p}^2+2y_{12}}{y_{1p}y_{2p}}\,|\mathcal{M}^{(0)}_2|^2. \tag{B.5}$$

Here $|\mathcal{M}^{(0)}_2|^2$ is the squared matrix element for the production of a quark-antiquark pair in electron-positron annihilation, averaged over event orientation (so it has no dependnece on parton momenta). Substituting eq. (B.5) into eq. (B.3), we find:

$$\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = -8\pi\alpha_{\mathrm{s}}\,C_{\mathrm{F}}\,\frac{1}{Q^2}\,|\mathcal{M}^{(0)}_2|^2\,\frac{1}{y_{1p}y_{2p}}\,T^{\mu\nu}\,. \tag{B.6}$$

The tree-level Altarelli-Parisi splitting kernel for $g\to gg$ splitting, $\langle\mu|\hat{P}^{(0)}_{gg}(\zeta,\kappa;\varepsilon)|\nu\rangle$ is given in eq. (3.5). The case of interest to us is when p in the spin-polarisation tensor is the collinear direction of the $g\to gg$ splitting. In particular this means that $p\cdot\kappa = 0$.

With the explicit expressions for $\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p)$ and $\langle\mu|\hat{P}^{(0)}_{gg}(\zeta,\kappa;\varepsilon)|\nu\rangle$, we can carry out the contraction

$$\begin{aligned} & \langle\mu|\hat{P}^{(0)}_{gg}(\zeta,\kappa;\varepsilon)|\nu\rangle\,\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = \\ & \quad = 2C_{\mathrm{A}}\Bigg\{|\mathcal{M}^{(0)}_3(p_1,p_2,p)|^2\,\left(\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}\right) + \\ & \qquad + 8\pi\alpha_{\mathrm{s}}C_{\mathrm{F}}\,|\mathcal{M}^{(0)}_2|^2\,\frac{1-\varepsilon}{s_{34}}\left[\left(\frac{s_{1\kappa}}{s_{1p}}-\frac{s_{2\kappa}}{s_{2p}}\right)^2 + \frac{s_{34}}{Q^2}\,\zeta(1-\zeta)\,\frac{s_{1p}^2+s_{2p}^2}{s_{1p}s_{2p}}\right]\Bigg\}, \end{aligned} \tag{B.7}$$

where we have introduced the following notation:

$$s_{i\kappa} = 2p_i\kappa, \quad s_{ip} = 2p_ip, \quad i=1,2\,, \qquad s_{34} = -\frac{\kappa^2}{\zeta(1-\zeta)}\,. \tag{B.8}$$

JHEP06(2005)024

We now derive the $p_1||p$ collinear limit of eq. (B.7), which is precisely defined by the following Sudakov parametrization of the momenta:

$$\begin{aligned} p_1^\mu &= z_1 P^\mu + k_\perp^\mu - \frac{k_\perp^2}{z_1}\frac{n^\mu}{2Pn}\,, \qquad p^\mu = (1-z_1)P^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z_1}\frac{n^\mu}{2Pn}\,, \\ s_{1p} &= -\frac{k_\perp^2}{z_1(1-z_1)} = \mathrm{O}(k_\perp^2)\,, \qquad k_\perp \to 0\,, \end{aligned} \tag{B.9}$$

where $Pk_\perp = nk_\perp = P^2 = n^2 = 0$. When calculating $s_{1\kappa}$, we use $p\kappa = 0$ to eliminate the $P\kappa$ term from $s_{1\kappa}$. Then we get

$$s_{1\kappa} = \frac{2}{1-z_1}k_\perp\cdot\kappa + 2\left(\frac{z_1}{(1-z_1)^2} - \frac{1}{z_1}\right)\frac{n\kappa}{2nP}k_\perp^2 = \mathrm{O}(k_\perp). \tag{B.10}$$

Then the collinear limit of eq. (B.7) reads

$$\begin{aligned} &\mathbf{C}_{1p}\langle\mu|\hat{P}^{(0)}_{gg}(\zeta,\kappa;\varepsilon)|\nu\rangle\,\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = \\ &\quad = 8\pi\alpha_\mathrm{s}\,|\mathcal{M}_2^{(0)}(P,p_2)|^2\,2C_\mathrm{A}\,\frac{1}{s_{1p}}\times \\ &\quad\times\left\{\left(\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}\right)P^{(0)}_{qg}(z_1;\varepsilon) + C_\mathrm{F}\,\frac{1-\varepsilon}{s_{34}}\left[\frac{s_{1\kappa}^2}{s_{1p}} + s_{34}\,\zeta(1-\zeta)\,(1-z_1)\right]\right\}. \end{aligned} \tag{B.11}$$

We can compute the same limit using the expression derived for the collinear limit of the spin-polarisation tensor in section 6.3. Defining the $p_1||p$ limit as in eq. (B.9), we have (see eq. (6.22)):

$$\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) \simeq 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\frac{1}{s_{1p}}\,\langle\mathcal{M}_2^{(0)}(P,p_2)|\,\hat{P}^{\mu\nu}_{gq}(z_1,k_\perp;P,n,\varepsilon)\,|\mathcal{M}_2^{(0)}(P,p_2)\rangle\,, \tag{B.12}$$

where $\langle r|\hat{P}^{\mu\nu}_{gq}(\zeta,\kappa;p,n,\varepsilon)|s\rangle$ is given in eq. (6.31). Using eq. (B.12) we obtain for the collinear limit of eq. (B.7)

$$\begin{aligned} &\mathbf{C}_{1p}\,\langle\mu|\hat{P}^{(0)}_{gg}(\zeta,\kappa;\varepsilon)|\nu\rangle\,\mathcal{T}^{\mu\nu}_{q\bar{q}g}(p_1,p_2,p) = \\ &\quad = 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\,|\mathcal{M}_2^{(0)}(P,p_2)|^2\,2C_\mathrm{A}\,\frac{1}{s_{1p}}\times \\ &\quad\times\left\{\left(\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}\right)P^{(0)}_{qg}(z_1;\varepsilon) + \right. \\ &\qquad\left. + 2C_\mathrm{F}\,\frac{1-\varepsilon}{s_{34}}\left[-\frac{1-z_1}{2}\kappa^2 - 2\frac{z_1}{1-z_1}\frac{(k_\perp\cdot\kappa)^2}{k_\perp^2} + (1-z_1)\frac{(P\kappa)(n\kappa)}{Pn}\right]\right\}. \end{aligned} \tag{B.13}$$

To see that eqs. (B.11) and (B.13) are in fact equal up to subleading terms, note the following. From $p\kappa = 0$, we get

$$P\kappa = \frac{1}{1-z_1}k_\perp\cdot\kappa + \frac{z_1 k_\perp^2}{(1-z_1)^2}\frac{n\kappa}{2Pn} = \mathrm{O}(k_\perp)\,, \tag{B.14}$$

so the last term in the square brackets in eq. (B.13) is subleading. Furthermore, we see from eqs. (B.9) and (B.10) that

$$\frac{(k_\perp\cdot\kappa)^2}{k_\perp^2} = -\frac{1-z_1}{4z_1}\frac{s_{1\kappa}^2}{s_{1p}} + \mathrm{O}(k_\perp)\,. \tag{B.15}$$

Using eqs. (B.14) and (B.15) in eq. (B.13) we obtain eq. (B.11). This gives a direct check of result eq. (6.31) for the splitting tensor.

JHEP06(2005)024

C. The collinear limit of the spin-polarization tensors using a helicity basis

In this appendix we record the collinear limit of the $\mathcal{T}^{hh'}$ tensors using a helicity basis. There are two ways to find the factorization formula. One is to work directly with helicity amplitudes and use the known factorization formulae with the splitting functions factorized. The other is to project the factorization formulae of section 6.3 onto a helicity basis. We explore both ways and show that these lead to the same expressions. The splitting functions are defined in $d = 4$ dimensions, therefore, in this appendix we set $\varepsilon = 0$ everywhere.

Let us start with the former. In our notation the helicity amplitudes are obtained as

$$\mathcal{A}_m^{\text{tree}}(1^{\lambda_1}, \ldots, m^{\lambda_m}) = \langle \lambda_1, \ldots, \lambda_m | \mathcal{M}_m^{(0)}(p_1, \ldots, p_m) \rangle \,. \tag{C.1}$$

The helicity amplitudes can be written as a sum of products of colour factors C_σ and partial amplitudes A_{m+1}^{tree},

$$\mathcal{A}_m^{\text{tree}}(1^{\lambda_1}, \ldots, m^{\lambda_m}) = g_{\text{s}}^{m-2} \sum_\sigma C_{\sigma(\{1,\ldots,m\})} A_m^{\text{tree}}\left(\sigma(\{1^{\lambda_1}, \ldots, m^{\lambda_m}\})\right) , \tag{C.2}$$

where $\sigma(\{1, \ldots, m\})$ denotes a certain permutation of the labels. The allowed permutations are not important in our present calculation.

The collinear-factorization formula for the partial amplitudes when partons i and r become collinear reads (see e.g. ref. [41])

$$A_{m+1}^{\text{tree}}(i^{\lambda_i}, r^{\lambda_r}, \ldots) = \text{Split}_{-\lambda}^{\text{tree}}(i^{\lambda_i}, r^{\lambda_r}) A_m^{\text{tree}}(p_{ir}^{\lambda}, \ldots) + \cdots \,, \tag{C.3}$$

where the ellipses mean subleading terms, which are neglected in the collinear limit. Substituting eq. (C.3) into the definition of the helicity-dependent tensor

$$\mathcal{T}_{m+1}^{\lambda_i \lambda_i'}(p_i, p_r, \ldots) = \langle \mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots) | \lambda_i \rangle \langle \lambda_i' | \mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots) \rangle \,, \tag{C.4}$$

we obtain the factorization formula

$$\mathbf{C}_{ir} \mathcal{T}_{m+1}^{\lambda_i \lambda_i'}(p_i, p_r, \ldots) = 8\pi\alpha_{\text{s}}\, \boldsymbol{T}_{ir}^2 \sum_{\lambda\lambda'} \langle \lambda | \hat{P}_{g_i f_r}^{\lambda_i \lambda_i'}(z_r, \langle ir \rangle) | \lambda' \rangle \mathcal{A}_m^{\text{tree}}(p_{ir}^{\lambda}, \ldots) \left[\mathcal{A}_m^{\text{tree}}(p_{ir}^{\lambda'}, \ldots) \right]^* \tag{C.5}$$

where

$$\langle \lambda | \hat{P}_{f_i f_r}^{\lambda_i \lambda_i'}(z_r, \langle ir \rangle) | \lambda' \rangle = \frac{1}{2} \sum_{\lambda_r \lambda_r'} \text{Split}_{-\lambda}^{\text{tree}}(i^{\lambda_i}, r^{\lambda_r}) \left[\text{Split}_{-\lambda'}^{\text{tree}}(i^{\lambda_i'}, r^{\lambda_r'}) \right]^* \tag{C.6}$$

is the representation of the $\hat{P}_{f_i f_r}^{\alpha\beta}$ splitting tensors on a helicity basis and $\langle ir \rangle$ is the spinor product as defined, e.g., in ref. [41]. Using the known splitting functions for the processes $q \to q_r + g_i$ and $g \to g_r + g_r$, we can easily compute the tensors $\langle \lambda | \hat{P}_{g_i q_r}^{\lambda_i \lambda_i'} | \lambda' \rangle$ and $\langle \lambda | \hat{P}_{g_i g_r}^{\lambda_i \lambda_i'} | \lambda' \rangle$. For the case of quark splitting we have

$$\langle \lambda | \hat{P}_{g_i q_r}^{\lambda_i \lambda_i'} | \lambda' \rangle = \delta_{\lambda\lambda'} P_{g_i q_r}^{\lambda_i \lambda_i'} \,, \tag{C.7}$$

JHEP06(2005)024

therefore, we have to compute only the following cases,

$$
\begin{aligned}
P^{++}_{g_i q_r} &= P^{--}_{g_i q_r} = \frac{1}{s_{ir}} \frac{1+z^2}{2(1-z)} = \frac{1}{s_{ir}} \left(\frac{z}{1-z} + \frac{1-z}{2} \right) , \\
P^{+-}_{g_i q_r} &= \left(P^{-+}_{g_i q_r} \right)^* = \frac{1}{\langle ir \rangle^2} \frac{z}{1-z} .
\end{aligned}
\tag{C.8}
$$

For the gluon splitting we find

$$
\begin{aligned}
\langle + | \hat{P}^{++}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle &= \langle - | \hat{P}^{--}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle = \frac{1}{s_{ir}} \frac{z^3}{1-z} , \\
\langle + | \hat{P}^{+-}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle &= \langle - | \hat{P}^{-+}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle = 0 , \\
\langle + | \hat{P}^{--}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle &= \langle - | \hat{P}^{++}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle = \frac{1}{s_{ir}} \left(\frac{(1-z)^3}{z} + \frac{1}{z(1-z)} \right) , \\
\langle + | \hat{P}^{-+}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle &= \langle - | \hat{P}^{+-}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle = \frac{2}{s_{ir}} \frac{1-z}{z} , \\
\langle + | \hat{P}^{+-}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle &= \langle - | \hat{P}^{+-}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle = \langle + | \hat{P}^{-+}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle^* = \\
&= \langle - | \hat{P}^{-+}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle^* = \frac{1}{\langle ir \rangle^2} \frac{z}{1-z} , \\
\langle + | \hat{P}^{++}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle &= \langle + | \hat{P}^{--}_{g_i g_r}(z_r, \langle ir \rangle) | - \rangle = \langle - | \hat{P}^{++}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle^* = \\
&= \langle - | \hat{P}^{--}_{g_i g_r}(z_r, \langle ir \rangle) | + \rangle^* = \frac{1}{\langle ir \rangle^2} z(1-z) .
\end{aligned}
\tag{C.9}
$$

We can derive the factorization formula in eq. (C.5) from eq. (6.22) by inserting the resolution of the identity in terms of projection operators,

$$
\sum_\lambda | \lambda \rangle \langle \lambda | = 1 , \tag{C.10}
$$

and projecting the Lorentz indices onto helicity states with polarization vectors $\varepsilon^\lambda_\mu \equiv \langle \lambda | \mu \rangle$,

$$
\begin{aligned}
&\mathbf{C}_{ir} \mathcal{T}^{\lambda_i \lambda'_i}_{m+1}(p_i, p_r, \ldots) = \\
&\quad = 8\pi\alpha_{\mathrm{s}} \frac{1}{s_{ir}} \sum_{\lambda\lambda'} \langle \mathcal{M}^{(0)}_m(p_{ir}, \ldots) | \lambda \rangle \, \langle \lambda | \mu \rangle \, \times \\
&\qquad \times \langle \lambda_i | \alpha \rangle \, \langle \mu | \hat{P}^{\alpha\beta}_{g_i f_r}(z_r, k_{r\perp}; p_{ir}, n_{ir}) \rangle | \nu \rangle \, \langle \beta | \lambda'_i \rangle \, \langle \nu | \lambda' \rangle \, \langle \lambda' | \mathcal{M}^{(0)}_m(p_{ir}, \ldots) \rangle .
\end{aligned}
\tag{C.11}
$$

Comparing it to eq. (C.5), we see that

$$
\boldsymbol{T}^2_{ir} \, \langle \lambda | \hat{P}^{\lambda_i \lambda'_i}_{g_i f_r}(z_r, \langle ir \rangle) | \lambda' \rangle = \frac{1}{s_{ir}} \langle \lambda | \mu \rangle \langle \lambda_i | \alpha \rangle \, \langle \mu | \hat{P}^{\alpha\beta}_{g_i f_r}(z_r, k_{r\perp}; p_{ir}, n_{ir}) \rangle | \nu \rangle \, \langle \beta | \lambda'_i \rangle \, \langle \nu | \lambda' \rangle . \tag{C.12}
$$

In order to compute the right-hand side of the equation above from eqs. (6.31) and (6.32), we note first that the gauge terms do not contribute. Secondly, in eq. (6.32) we left out terms that do not contribute to either eqs. (6.25) and (6.26) or eq. (6.33). Including these terms and neglecting the gauge ones, we have

$$
\langle s | \hat{P}^{\alpha\beta}_{gq}(z, k_\perp; p, n; \varepsilon) | s' \rangle = C_{\mathrm{F}} \, \delta_{ss'} \left[-\frac{1-z}{2} g^{\alpha\beta} - 2 \frac{z}{1-z} \frac{k^\alpha_\perp k^\beta_\perp}{k^2_\perp} \right] + \cdots , \tag{C.13}
$$

JHEP06(2005)024

$$
\begin{aligned}
\langle\mu|\hat{P}^{\alpha\beta}_{gg}(z,k_\perp;p,n;\varepsilon)|\nu\rangle \;=\; & 2C_{\mathrm{A}}\left[\frac{1-z}{z}g^{\alpha\mu}g^{\beta\nu}+\frac{z}{1-z}g^{\mu\nu}\frac{k_\perp^\alpha k_\perp^\beta}{k_\perp^2}+z(1-z)g^{\alpha\beta}\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}+\right.\\
& \left.+\frac{z}{k_\perp^2}\left(g^{\mu\alpha}k_\perp^\beta k_\perp^\nu+g^{\beta\nu}k_\perp^\mu k_\perp^\alpha-g^{\alpha\nu}k_\perp^\mu k_\perp^\beta-g^{\beta\mu}k_\perp^\nu k_\perp^\alpha\right)\right]+\\
& +\cdots\,.
\end{aligned}
\tag{C.14}
$$

Using the conventions for spinor products and polarization vectors of ref. [41] and the representation of the transverse momenta in eq. (6.56), we can derive the following relations, valid in the collinear limit,

$$
\langle\lambda|\mu\rangle\, g^{\mu\nu}\,\langle\nu|\lambda'\rangle=-\delta_{\lambda\lambda'}\,,\qquad \langle\lambda|\mu\rangle\,\langle\lambda'|\alpha\rangle\, g^{\mu\alpha}=-\delta_{\lambda(-\lambda')}\,,
\tag{C.15}
$$

$$
-2\,\langle\lambda|\mu\rangle\,\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\,\langle\nu|\lambda'\rangle=-2\,\langle\lambda|\alpha\rangle\,\frac{k_\perp^\alpha k_\perp^\beta}{k_\perp^2}\,\langle\beta|\lambda'\rangle=\delta_{\lambda\lambda'}+\delta_{\lambda(-\lambda')}\frac{s_{js}}{\langle js\rangle^2}
\tag{C.16}
$$

and

$$
-2\,\langle\lambda|\mu\rangle\,\langle-\lambda|\alpha\rangle\,\frac{k_\perp^\mu k_\perp^\alpha}{k_\perp^2}=1\,.
\tag{C.17}
$$

Using these relations when inserting eqs. (C.13) and (C.14) into eq. (C.12), we obtain the same results as in eqs. (C.8) and (C.9).

References

[1] W.T. Giele and E.W.N. Glover, *Higher order corrections to jet cross-sections in e^+e^- annihilation*, *Phys. Rev.* **D 46** (1992) 1980.

[2] W.T. Giele, E.W.N. Glover and D.A. Kosower, *Higher order corrections to jet cross-sections in hadron colliders*, *Nucl. Phys.* **B 403** (1993) 633 [hep-ph/9302225].

[3] S. Frixione, Z. Kunszt and A. Signer, *Three-jet cross sections to next-to-leading order*, *Nucl. Phys.* **B 467** (1996) 399 [hep-ph/9512328].

[4] Z. Nagy and Z. Trócsányi, *Calculation of QCD jet cross sections at next-to-leading order*, *Nucl. Phys.* **B 486** (1997) 189 [hep-ph/9610498].

[5] S. Frixione, *A general approach to jet cross sections in QCD*, *Nucl. Phys.* **B 507** (1997) 295 [hep-ph/9706545].

[6] S. Catani and M.H. Seymour, *A general algorithm for calculating jet cross sections in NLO QCD*, *Nucl. Phys.* **B 485** (1997) 291 [hep-ph/9605323].

[7] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three-loop splitting functions in QCD: the non-singlet case*, *Nucl. Phys.* **B 688** (2004) 101 [hep-ph/0403192];
A. Vogt, S. Moch and J.A.M. Vermaseren, *The three-loop splitting functions in QCD: the singlet case*, *Nucl. Phys.* **B 691** (2004) 129 [hep-ph/0404111].

[8] R. Hamberg, W.L. van Neerven and T. Matsuura, *A complete calculation of the order α_s^2 correction to the Drell-Yan K factor*, *Nucl. Phys.* **B 359** (1991) 343.

[9] R.V. Harlander and W.B. Kilgore, *Next-to-next-to-leading order Higgs production at hadron colliders*, *Phys. Rev. Lett.* **88** (2002) 201801 [hep-ph/0201206].

JHEP06(2005)024

[10] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, *Dilepton rapidity distribution in the Drell-Yan process at NNLO in QCD*, *Phys. Rev. Lett.* **91** (2003) 182002 [hep-ph/0306192].

[11] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, *High-precision QCD at hadron colliders: electroweak gauge boson rapidity distributions at NNLO*, *Phys. Rev.* **D 69** (2004) 094008 [hep-ph/0312266].

[12] C. Anastasiou and K. Melnikov, *Higgs boson production at hadron colliders in NNLO QCD*, *Nucl. Phys.* **B 646** (2002) 220 [hep-ph/0207004].

[13] C. Anastasiou, K. Melnikov and F. Petriello, *Higgs boson production at hadron colliders: differential cross sections through next-to-next-to-leading order*, hep-ph/0409088.

[14] C. Anastasiou, K. Melnikov and F. Petriello, *Fully differential Higgs boson production and the di-photon signal through next-to-next-to-leading order*, hep-ph/0501130.

[15] C. Anastasiou, K. Melnikov and F. Petriello, *Real radiation at nnlo: $e^+e^- \to 2$ jets through $O(\alpha_s^2)$*, *Phys. Rev. Lett.* **93** (2004) 032002 [hep-ph/0402280].

[16] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Infrared structure of $e^+e^- \to 2$ jets at NNLO*, *Nucl. Phys.* **B 691** (2004) 195 [hep-ph/0403057].

[17] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Infrared structure of $e^+e^- \to$ 3jets at NNLO: the C_f^2 contribution*, *Nucl. Phys.* **135** *(Proc. Suppl.)* (2004) 97 [hep-ph/0407023].

[18] G. Heinrich, *A numerical method for NNLO calculations*, *Nucl. Phys.* **116** *(Proc. Suppl.)* (2003) 368 [hep-ph/0211144].

[19] C. Anastasiou, K. Melnikov and F. Petriello, *A new method for real radiation at NNLO*, *Phys. Rev.* **D 69** (2004) 076010 [hep-ph/0311311].

[20] T. Binoth and G. Heinrich, *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl. Phys.* **B 693** (2004) 134 [hep-ph/0402265].

[21] G. Heinrich, *A numerical approach to infrared divergent multi-parton phase space integrals*, *Nucl. Phys.* **135** *(Proc. Suppl.)* (2004) 290 [hep-ph/0406332].

[22] D.A. Kosower, *Antenna factorization of gauge-theory amplitudes*, *Phys. Rev.* **D 57** (1998) 5410 [hep-ph/9710213].

[23] J.M. Campbell, M.A. Cullen and E.W.N. Glover, *Four jet event shapes in electron positron annihilation*, *Eur. Phys. J.* **C 9** (1999) 245 [hep-ph/9809429].

[24] S. Weinzierl, *Subtraction terms at NNLO*, *JHEP* **03** (2003) 062 [hep-ph/0302180].

[25] S. Weinzierl, *Subtraction terms for one-loop amplitudes with one unresolved parton*, *JHEP* **07** (2003) 052 [hep-ph/0306248].

[26] D.A. Kosower, *Antenna factorization in strongly-ordered limits*, *Phys. Rev.* **D 71** (2005) 045016 [hep-ph/0311272].

[27] A. Gehrmann-De Ridder, T. Gehrmann and G. Heinrich, *Four-particle phase space integrals in massless QCD*, *Nucl. Phys.* **B 682** (2004) 265 [hep-ph/0311276].

[28] S. Frixione and M. Grazzini, *Subtraction at NNLO*, hep-ph/0411399.

[29] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Quark-gluon antenna functions from neutralino decay*, *Phys. Lett.* **B 612** (2005) 36 [hep-ph/0501291].

JHEP06(2005)024

[30] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, *Gluon gluon antenna functions from Higgs boson decay*, *Phys. Lett.* **B 612** (2005) 49 [hep-ph/0502110].

[31] J.M. Campbell and E.W.N. Glover, *Double unresolved approximations to multiparton scattering amplitudes*, *Nucl. Phys.* **B 527** (1998) 264 [hep-ph/9710255].

[32] S. Catani and M. Grazzini, *Collinear factorization and splitting functions for next-to-next-to-leading order QCD calculations*, *Phys. Lett.* **B 446** (1999) 143 [hep-ph/9810389].

[33] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys.* **B 568** (2000) 211 [hep-ph/9909464].

[34] D.A. Kosower, *Multiple singular emission in gauge theories*, *Phys. Rev.* **D 67** (2003) 116003 [hep-ph/0212097].

[35] A. Bassetto, M. Ciafaloni and G. Marchesini, *Jet structure and infrared sensitive quantities in perturbative QCD*, *Phys. Rept.* **100** (1983) 201.

[36] G. Altarelli and G. Parisi, *Asymptotic freedom in parton language*, *Nucl. Phys.* **B 126** (1977) 298.

[37] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl. Phys.* **B 570** (2000) 287 [hep-ph/9908523].

[38] R.K. Ellis, D.A. Ross and A.E. Terrano, *The perturbative calculation of jet structure in* e^+e^- *annihilation*, *Nucl. Phys.* **B 178** (1981) 421.

[39] S. Catani and M. Grazzini, *The soft-gluon current at one-loop order*, *Nucl. Phys.* **B 591** (2000) 435 [hep-ph/0007142].

[40] Z. Nagy and Z. Trócsányi, *Next-to-leading order calculation of four-jet observables in electron positron annihilation*, *Phys. Rev.* **D 59** (1999) 014020 [hep-ph/9806317].

[41] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301.

JHEP06(2005)024